HIGH SCHOOL MATHEMATICS

Unit 5.

RELATIONS AND FUNCTIONS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1960

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TABLE OF CONTENTS

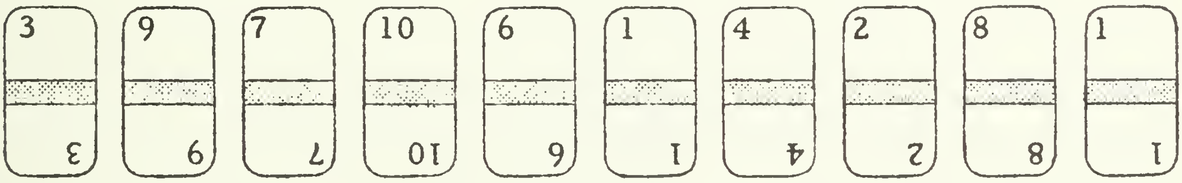
<u>The Games Club</u>.--Morris Thompson, a new student at Zabranchburg High School, is visiting a regular meeting of the Games Club. The president announces:

Today we play TREE.

The members form pairs, and Tony invites Morris to be his opponent. Tony starts to explain the card game TREE to him, but Morris says, "Let's just play a few hands; I'm sure I'll catch on that way."

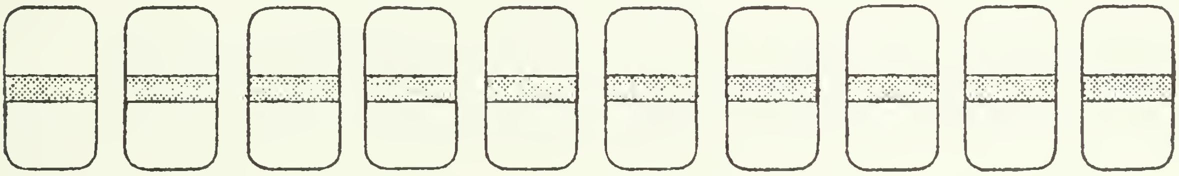
So, Tony hands him a chart which looks like the one on page 5-B. Then, he picks up a deck of twenty cards, shuffles it, and deals ten cards to Morris and ten cards to himself. Morris picks up his cards.

<u>Morris' cards</u>

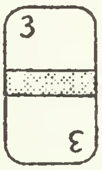


Since the deck is made up of of twenty cards, two 1s, two 2s, two 3s, etc., you can tell what Tony's cards are. So, fill in the numerals:

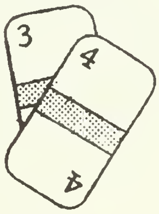
<u>Tony's cards</u>



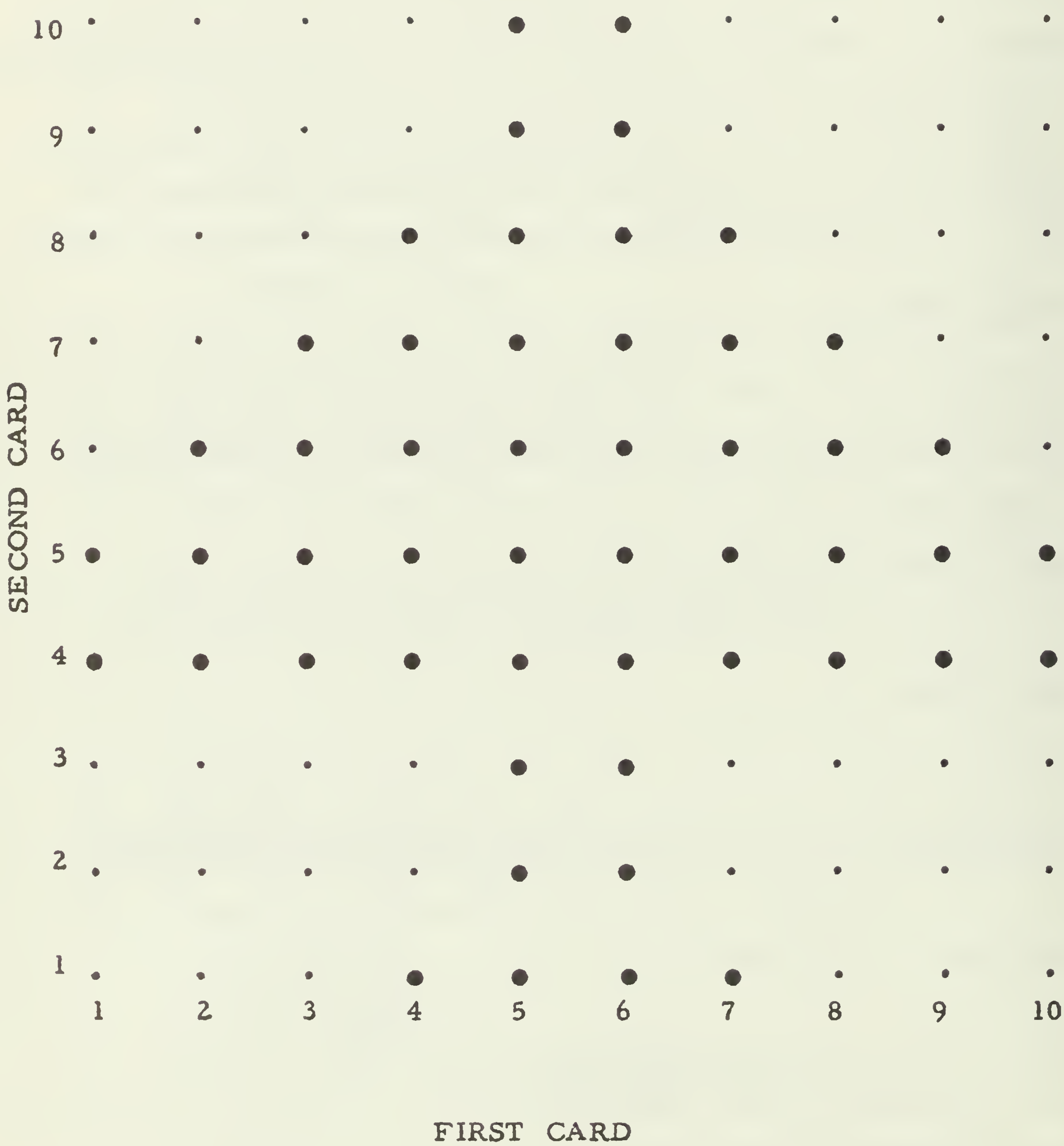
Tony says, "I'll play first," and plays a



Morris, thinking that this game is like most card games, plays a higher card, a 4, and reaches for the cards.



TREE CHART

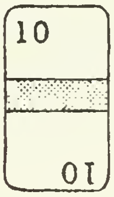


Tony, after glancing at the chart, says, "O.K., you win that trick. 3 loses to 4 because

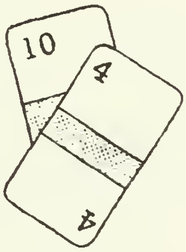
4 TREES 3.

Now, you play first because you won."

So, Morris plays as first card a



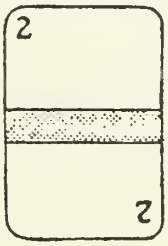
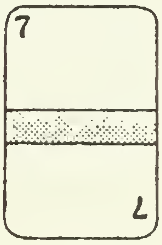
and Tony plays as second card a 4.

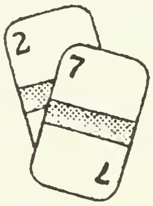


Morris, thinking he has won again, starts to reach for the pair of cards but Tony stops him. "No, Morris, look at the chart; I win this trick." 10 loses to 4 because

4 TREES 10

Morris is puzzled, but they continue to play with Tony telling the winner of each trick after looking at the chart. Here are their next few plays.

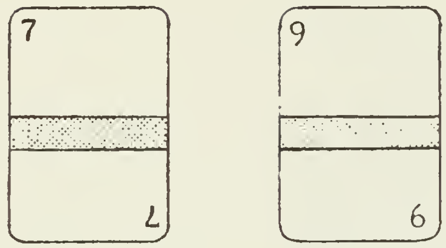
	Tony 1st Card	Morris 2nd Card	
Tony first			1st Card wins because 7 DOES NOT TREE 2.



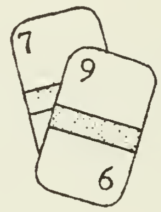
Tony first

Tony
1st Card

Morris
2nd Card



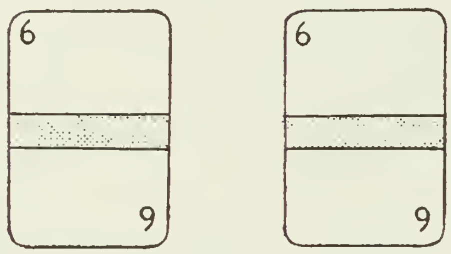
1st Card wins because
9 DOES NOT TREE 7.



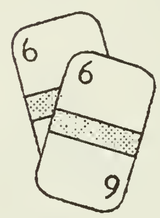
Tony first

Tony
1st Card

Morris
2nd Card



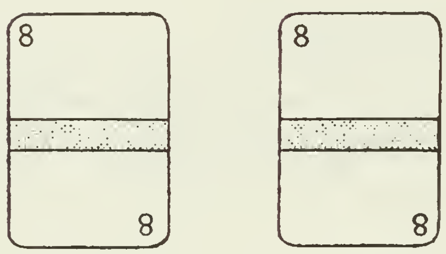
2nd Card wins because
6 TREES 6.



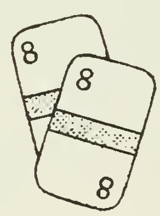
Morris first

Morris
1st Card

Tony
2nd Card



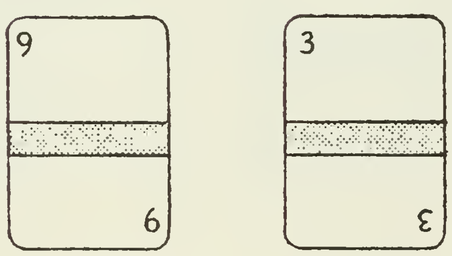
1st Card wins because
8 DOES NOT TREE 8.



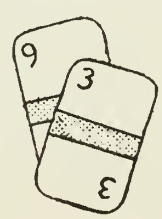
Morris first

Morris
1st Card

Tony
2nd Card



1st Card wins because
3 DOES NOT TREE 9.



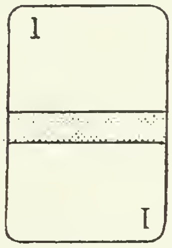
EXERCISES

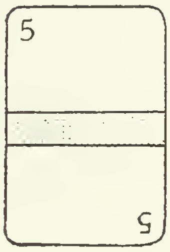
A. Who wins in each of these last three plays? Fill in the blanks at the right.

Morris
1st Card

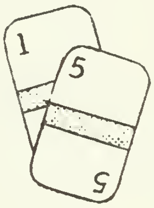
Tony
2nd Card

Morris first





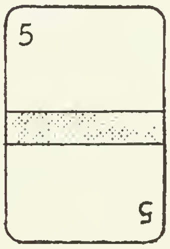
___ Card wins because
5 ________________ 1.

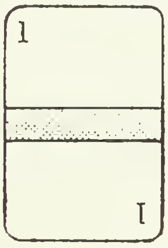


Tony
1st Card

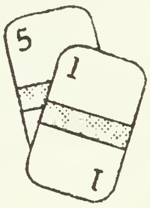
Morris
2nd Card

Tony first





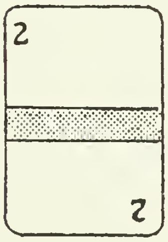
___ Card wins because
1 ________________ 5.

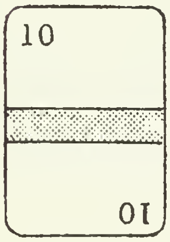


Morris
1st Card

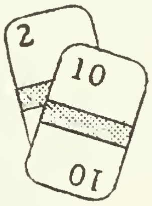
Tony
2nd Card

Morris first





___ Card wins because
10 ________________ 2.



B. By now Morris has caught on to the game and sees how to use the chart in order to tell the winner of a trick, that is, in order to tell when a second card TREES a first card. He sees that

4 TREES 6 and 6 TREES 4,
1 TREES 6 but 6 DOES NOT TREE 1,
3 DOES NOT TREE 8 and 8 DOES NOT TREE 3.

Use the TREE chart on page 5-B to tell which of the following are true statements and which are false statements.

1. 4 TREES 5	2. 7 TREES 3	3. 3 TREES 3
4. 2 TREES 6	5. 10 TREES 1	6. 1 TREES 10
7. 7 TREES 7	8. 5 TREES 10	9. 10 TREES 5
10. 6 DOES NOT TREE 10		11. 4 DOES NOT TREE 4
12. 7 DOES NOT TREE 5		13. 1 DOES NOT TREE 1

* * *

The chart on page 5-B pictures a 10-by-10 lattice. This lattice is the set of ordered pairs which is the cartesian square $D \times D$, where $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ['D' for 'decade'.] The chart for TREE pictures a subset of $D \times D$. Let's call this subset 'T'. The set of ordered pairs in $D \times D$ which do not belong to T is called <u>the complement of T</u> [with respect to $D \times D$]. 'the complement of T' is commonly abbreviated '$\tilde{T}$'. What is $T \cup \tilde{T}$? What is $T \cap \tilde{T}$?

In order to tell whether the sentence:

4 TREES 5

is true, you can look at the TREE chart and see whether

(5, 4) is an element of T.

In other words [abbreviating 'is an element of' by 'ϵ'], the sentences:

4 TREES 5 and: $(5, 4) \epsilon T$

are equivalent sentences. Do you see that

'4 DOES NOT TREE 9' and '$(9, 4) \epsilon \tilde{T}$'

are equivalent sentences?

* * *

C. True or false?

1. (10, 1) is an element of T
2. (10, 1) is a member of $\tilde{T}$
3. (6, 9) belongs to T
4. (6, 9) belongs to $\tilde{T}$
5. $(7, 3) \in T$
6. $(4, 2) \in \tilde{T}$
7. $(5, 10) \in \tilde{T}$
8. $(9, 6) \in \tilde{T}$
9. $(2, 9) \in \tilde{T}$
10. $(6, 6) \in T$
11. $(8, 8) \in \tilde{T}$
12. $(2, 1) \in T$

D. Find the solution set of each of the following sentences.

Sample. $(x, 8) \in T$

Solution. Since $(4, 8) \in T$, $(5, 8) \in T$, $(6, 8) \in T$, and $(7, 8) \in T$, and no other pair in $D \times D$ with second component 8 belongs to T, the solution set of the sentence '$(x, 8) \in T$' is $\{4, 5, 6, 7\}$.

[Note. In some of the exercises it will be the case that no matter what numeral you substitute for 'x', you will not get a true sentence. This is like having an equation with no roots. In such cases the solution set is the empty set, and when you write your answer, just write: $\emptyset$.]

1. $(x, 3) \in T$ [Answer: $\{5, 6\}$]
2. $(x, 3) \in \tilde{T}$
3. $(x, 6) \in T$
4. $(x, 6) \in \tilde{T}$
5. $(6, x) \in T$
6. $(6, x) \in \tilde{T}$
7. $(x, 4) \in T$
8. $(4, x) \in T$
9. $(x, 2x) \in T$
10. $(2x, x) \in T$
11. $(x, x - 2) \in T$
12. $(x, x + 2) \in \tilde{T}$
13. $(x, 2x - 1) \in T$
14. $(x, 7 - x) \in T$
15. $(x, 7) \in \tilde{\tilde{T}}$
16. $(1, x) \in \tilde{\tilde{T}}$

* * *

You may recall from an earlier unit a quick way of writing a name for the solution set of a sentence. For example, take the sentence:

$$(x, 3) \in T$$

We can name the solution set of this sentence by:

$$\{x \in D: (x, 3) \in T\}$$

This name is read as 'the set of all x in D such that $(x, 3) \in T$'.

* * *

E. True or false?

Sample 1. $\{x \in D: (x, 3) \in T\} = \{x \in D: (x, 8) \in T\}$

Solution. One way to decide whether this sentence is true or false is to list the elements in the sets.

$$\{x \in D: (x, 3) \in T\} = \{5, 6\}$$
$$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}$$

Since there is at least one element in $\{4, 5, 6, 7\}$ which is not a member of $\{5, 6\}$, we conclude that

$$\{5, 6\} \neq \{4, 5, 6, 7\}.$$

So, the given sentence is false.

Sample 2. $\{x \in D: (x, 3) \in T\} \subseteq \{x \in D: (x, 8) \in T\}$

Solution. We know, from Sample 1, that

$$\{x \in D: (x, 3) \in T\} = \{5, 6\}$$
and $$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}.$$

Since there is no member of $\{5, 6\}$ which is not a member of $\{4, 5, 6, 7\}$, the first set is a subset of the second.

$$\{5, 6\} \subseteq \{4, 5, 6, 7\}.$$

So, the given sentence is true.

1. $\{x \in D: (x, 3) \in T\} = \{x \in D: (x, 2) \in T\}$
2. $\{x \in D: (x, 8) \in T\} \subseteq \{x \in D: (x, 3) \in T\}$
3. $\{x \in D: (x, 3) \in T\} \subset \{x \in D: (x, 8) \in T\}$

4. $\{x \in D: (x, 3) \in T\} \subseteq \{x \in D: (x, 2) \in T\}$

5. $\{x \in D: (x, 2) \in T\} \subseteq \{x \in D: (x, 3) \in T\}$

6. $\{x \in D: (x, 2) \in \tilde{T}\} = \{x \in D: (x, 3) \in \tilde{T}\}$

7. $\{x \in D: (x, 8) \in T\} = \{x \in D: (3, x) \in T\}$

8. $\{x \in D: (4, x) \notin T\} = \{x \in D: (7, x) \notin T\}$

9. $\{x \in D: (4, x) \in \tilde{T}\} \subseteq \{x \in D: (7, x) \in \tilde{T}\}$

10. $\{x \in D: (6, x) \in \tilde{T}\} \subseteq \{x \in D: (x, 4) \in \tilde{T}\}$

F. Simplify.

Sample 1. Simplify: $\{x \in D: (x, 2) \in T\} \cap \{x \in D: (x, 8) \in T\}$

Solution. We are interested in the intersection of the sets, that is, we want to know that elements they have in common. A way to find this out is to list the elements in the sets.

$$\{x \in D: (x, 2) \in T\} = \{5, 6\},$$
$$\{x \in D: (x, 8) \in T\} = \{4, 5, 6, 7\}$$

Since 5 belongs to each set and 6 belongs to each set, and since these are the only elements which belong to both, it follows that $\{5, 6\} \cap \{4, 5, 6, 7\} = \{5, 6\}$. So, '$\{x \in D: (x, 2) \in T\} \cap \{x \in D: (x, 8) \in T\}$' simplifies to '$\{5, 6\}$'.

1. $\{x \in D: (x, 5) \in T\} \cap \{x \in D: (x, 7) \in T\}$

2. $\{x \in D: (x, 5) \in T\} \cup \{x \in D: (x, 7) \in T\}$

3. $\{x \in D: (4, x) \in T\} \cap \{x \in D: (9, x) \in T\}$

4. $\{m \in D: (4, m) \notin T\} \cup \{m \in D: (9, m) \notin T\}$

5. $\{x \in D: (x, 10) \in T\} \cap \{x \in D: (1, x) \in T\}$

6. $\{a \in D: (7, a) \in \tilde{T}\} \cup \{b \in D: (3, b) \in T\}$

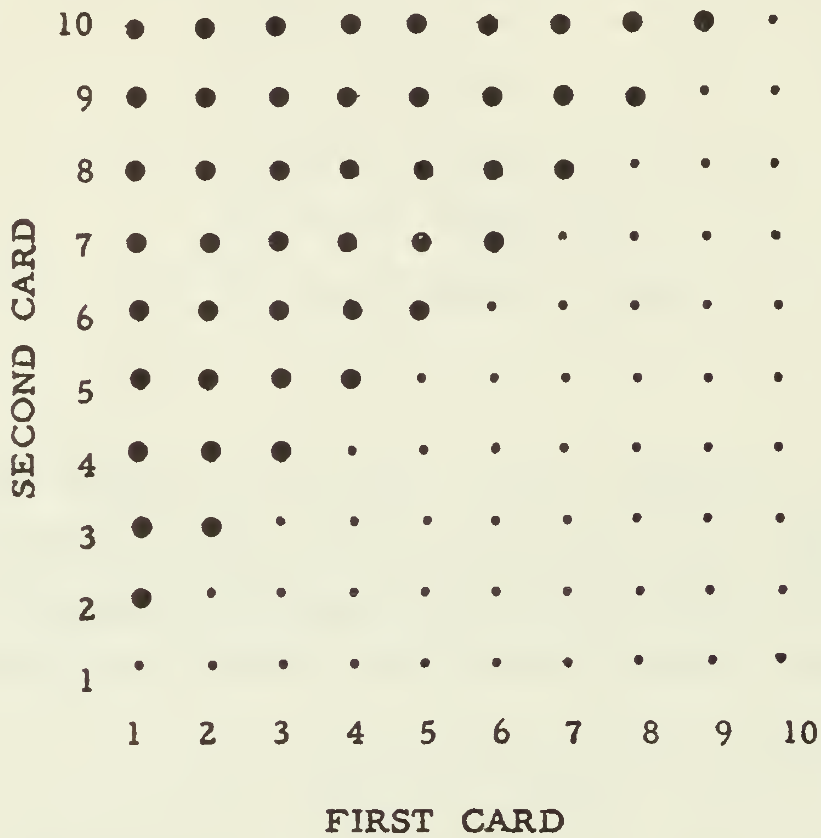
7. $\{x \in D: (8, x) \in \tilde{T}\} \cap \{x \in D: (x, 3) \in T\}$

8. $\{k \in D: (5, k) \in \tilde{T}\} \cap \{k \in D: (k, 4) \in T\}$

9. $\{n \in D: (n, 6) \in T\} \cup \{n \in D: (6, n) \in \tilde{T}\}$

10. $\{x \in D: (2, x) \in T\} \cap [\{x \in D: (x, 9) \in T\} \cup \{x \in D: (10, x) \in T\}]$

G. Another game which is played with the same cards as TREE is UPPER TRIANGLE. Here is the chart for UPPER TRIANGLE.



If the first player plays a 5 and the second player a 6, the second player wins the trick because (5, 6) corresponds with one of the heavy dots in the "upper triangle". For short, we say that

$$(5, 6) \in U.$$

If an 8 is played first and a 6 is played second, the 6 loses because $(8, 6) \in \tilde{U}$. What happens if each player plays a 3? Give the solution set of each of the following sentences.

1. $(x, 3) \in U$
2. $(x, 3) \in \tilde{U}$
3. $(10, x) \in U$
4. $(x, x) \in \tilde{U}$
5. $(x, 2x) \in U$
6. $(1, x) \in U$
7. $(6, x) \in U$
8. $(x, 7) \in \tilde{U}$
9. $(x, 3x - 2) \in U$
10. $(x, 4x - 7) \in U$
11. $(x, \frac{x}{2} + 1) \in \tilde{U}$
12. $(x, \frac{x}{2} - 1) \in \tilde{U}$

H. For each two games that the members of the Games Club invent, they can make up more games just by combining the charts for the two games. An example of such a new game is UPPER TRIANGLE INTERSECTION TREE. A chart for this game is a picture of $D \times D$ with the graph of $U \cap T$ drawn on it.

1. Make such a chart.

2. On another chart make a graph of $U \cup T$.

3. How many points are there in each of these sets?

(a) T (b) $\widetilde{T}$ (c) U

(d) $\widetilde{U}$ (e) $U \cup \widetilde{U}$ (f) $U \cap \widetilde{U}$

(g) $T \cup \widetilde{T}$ (h) $T \cap \widetilde{T}$ (i) $T \cap U$

(j) $\widetilde{T \cap U}$ (k) $\widetilde{T} \cup \widetilde{U}$ (ℓ) $T \cup U$

(m) $\widetilde{T \cup U}$ (n) $\widetilde{T} \cap \widetilde{U}$

I. 1. The experts in the Games Club play a game which differs from UPPER TRIANGLE only in that it is played with a deck of 200 cards, two 1s, two 2s, two 3s, etc. The set of ordered pairs in their "upper triangle" is a subset of a 100-by-100 lattice. [How many ordered pairs does <u>their</u> "upper triangle" contain?] Pick out the ordered pairs listed below which belong to the experts' UPPER TRIANGLE.

(a) (17, 4) (b) (31, 74) (c) (16, 17)

(d) (17, 16) (e) (16, 16) (f) (98, 1)

(g) (99, 100) (h) (61, 59) (i) (21, 3)

(j) (75, 76) (k) (42, 39) (ℓ) (81, 97)

2. The experts wanted a short name for the set of ordered pairs pictured on a chart for their game. They chose 'G'. Do you see why?

5.01 Relations.--One of the important aims of scientific work is to find answers to questions about how one thing is related to another. For example, an economist might want to know how lifetime income is related to years of formal education, or how wheat production is related to annual rainfall. A psychologist wants to know how the number of trials it takes a rat to learn a maze is related to the type of food the rat eats. A geneticist tries to describe the relation of eye color of children to eye color of parents. A physicist wants to express the relation of the range of a projectile to its initial velocity.

In asking such questions, the investigator is really trying to find out how one thing is affected by another. In order to discover this, the economist will, for example, collect data which he might tabulate like this:

Subject	Years of Schooling	Lifetime Income ($)
Mr. A	16	320,000
Miss B	14	200,000
Mr. C	8	150,000
Mrs. D	16	250,000
.	.	.
.	.	.
.	.	.

In other words, he collects ordered pairs [(16, 320 000), (14, 200 000), etc.], and then in order to study the relationship of income to education [or, how education affects income], he attempts to describe the set of ordered pairs. People investigating other such problems involving relationships also list ordered pairs, and their study of relationships leads them to describe sets of ordered pairs.

This tie-up between studying a relationship and describing a set of ordered pairs suggests that a set of ordered pairs be called a relation.

Let's take a very simple example. Suppose someone claims that he has investigated a certain relationship among the numbers in $\{-3, -2, -1, 0, 1, 2\}$ and has found that

−3 has this relationship to −2

−1 bears this relationship to 0, and so do −2 and −3,

−1 is related in this way to 1,

−3 bears this relationship to −1, and so does −2,

−3, −2, and 0 all are related this way to 1,

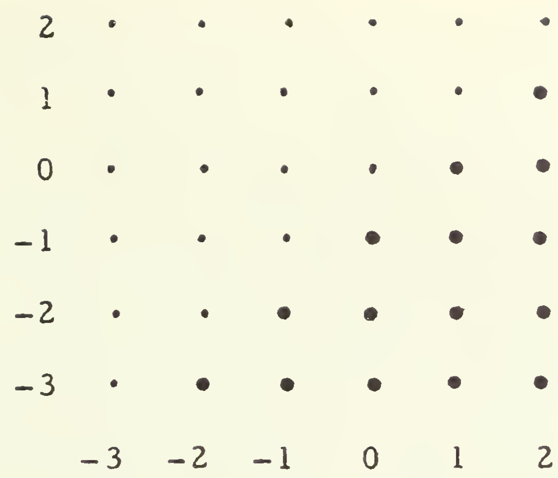
1, 0, −1, −2, and −3 all bear this relationship to 2.

You try to get more information; so, you ask, "Does 2 have this relationship to 0?" He answers, "No, it doesn't. I've told you about <u>all</u> the numbers in the set which have this relationship."

"Well, your description is a bit complicated. Perhaps I can follow it better if I record the facts in some kind of systematic way. This relationship you investigated is something which holds for ordered pairs of numbers in $\{-3, -2, -1, 0, 1, 2\}$. For the six numbers in the set, there are exactly ... um ..., yes, 6 times 6, or 36 ordered pairs.

2	•	•	•	•	•	•
1	•	•	•	•	•	•
0	•	•	•	•	•	•
−1	•	•	•	•	•	•
−2	•	•	•	•	•	•
−3	•	•	•	•	•	•
	−3	−2	−1	0	1	2

Now, I'll darken some of these dots. I go to the −3-column, to darken the dots corresponding to those numbers which have this relationship to −3. According to what you have told me, there are no such numbers. Next, I go to the −2-column. Since −3 is the only number which has this relationship to −2, I will darken the graph of $(-2, -3)$, and do nothing else to this column. Next, the −1-column. From your description, I darken just the graphs of $(-1, -3)$ and $(-1, -2)$. Continuing in this fashion, I get the following picture.



I have darkened a dot for each ordered pair of numbers whose second component has your relationship to its first component. So, the picture makes it easier for me to understand the relationship. I can tell at a glance, for example, that the only numbers to which -2 has this relationship are -1, 0, 1, and 2. And, now, I see an easy way to describe the <u>relation</u> whose graph I have drawn. It's the set of ordered pairs whose components belong to $\{-3, -2, -1, 0, 1, 2\}$ and such that the second component of each ordered pair is less than its first component. For short, it's

$$\{(x, y),\ x \text{ and } y \text{ integers from } -3 \text{ through } 2 : y < x\}.$$

If I use 'S', say, as a name for the set of integers from -3 through 2, and remember that the ordered pairs whose components both belong to S are just the members of $S \times S$, then I can write an even shorter name for this relation:

$$\{(x, y) \in S \times S:\ y < x\}$$

So, it's not as complicated as your description made it look."

There are many relations among the members of the set S. Some others, besides the one just discussed, are

$$\{(x, y) \in S \times S:\ y \neq x\},$$

$$\{(x, y) \in S \times S:\ y = x + 2\},$$

$$\{(x, y) \in S \times S:\ y = -1 \text{ or } x = 1\},$$

and

$$\{(-2, -1), (0, -2), (2, 1)\}.$$

Each relation among the members of S is a subset of $S \times S$; and, each subset of $S \times S$ is such a relation.

As you have seen, one way of describing a relation is, first, to tell about things you are interested in, and, then to list the ordered pairs of these things which belong to the relation. Instead of listing, you can use a graph. Suppose, for example, you were watching two members of the Games Club playing TREE, and that these players knew the game so well that they were playing it without a chart. In order to find out how the game is played, you might begin by noticing that the cards have values which belong to the set D of integers from 1 through 10. Then, you could keep track of the tricks played and who won each of them by starting a list like this:

(3, 4), second player

(10, 4), second player

(2, 7), first player

(7, 9), first player

(6, 6), second player

(8, 8), first player

•
•
•

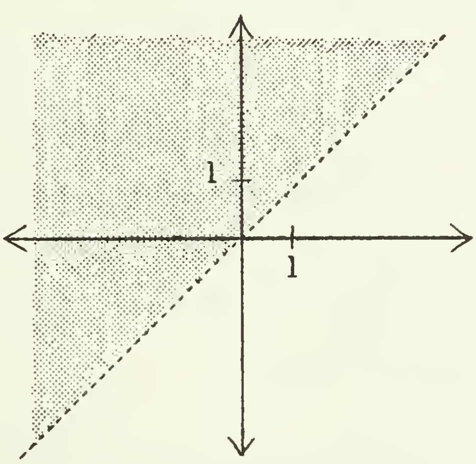
and continuing it for several games. You would probably find it difficult to discover any pattern from this list. So, you might graph the ordered pairs corresponding to tricks won by the second player, using a heavy dot for this purpose, and graph the tricks won by the first player, using light dots for these. If you kept this up long enough, you would get the picture shown on page 5-B. The heavy dots make up a graph of the relation T, a subset of $D \times D$. [The light dots make up a graph of the relation $\widetilde{T}$.] You have discovered that when cards with values x and y are played, in that order, the trick belongs to the second player if and only if $(x, y) \in T$. So, by discovering the relation T, you've learned how TREE is played.

NAMING A RELATION

As we have noted, a relation is a set--a set of ordered pairs. In order to talk about a relation we need a name for it, or a picture we can point to. It is usually convenient to have a name and, in the Introduction, we used the arbitrary names 'T' and 'U' [or 'G'] for the relations used in playing the games TREE and UPPER TRIANGLE. We could as well have called them 'Tom' and 'Ursula'. In either case, the name doesn't tell us anything about the relation. The relation has to be described in some way before we can know what the name means.

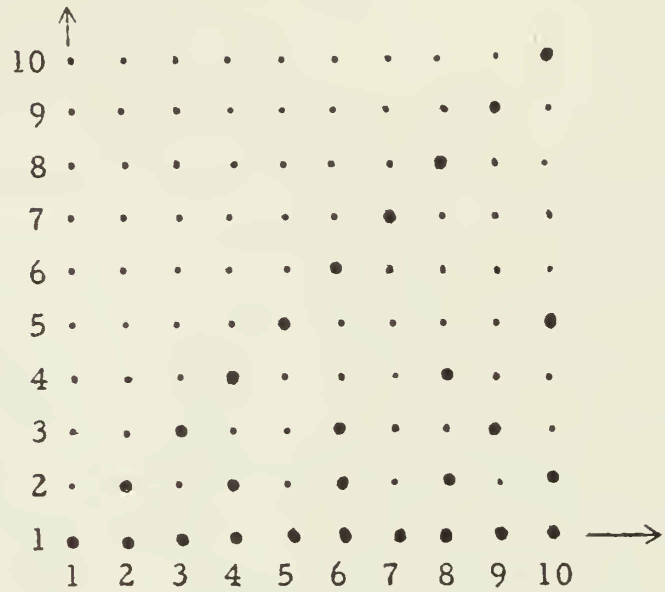
For example, we often have occasion to talk about the relation [among the real numbers] of being greater than. Since it is customary to abbreviate 'is greater than' by '>', mathematicians sometimes use the symbol '>' as a name for this relation. [Notice that in doing this one is using as a noun a symbol which is primarily used as a predicate. This could be confusing, but it is usually very easy to tell from context how the symbol is being used. For example, in the sentence '9 > 7', the '>' is a predicate; in the sentence '(7, 9) ϵ >', the '>' is a noun.] If you wanted to tell someone what '>' means, you would need a way of describing the relation. [Like 'T' and 'U', '>' is an arbitrary name and conveys no information until some kind of description is given.]

One way of describing the relation > is to draw a graph.



A person looking at the picture may feel fairly sure that he knows what '>' means. Of course, you would have to tell him that the picture is meant "to go on forever", and his main problem is what you mean by this. You may feel that this is not much of a problem, but that is because you are already well acquainted with the relation >.

However, consider the relation which mathematicians name by '|'. Here is a picture of part of the relation |.



This picture, too, is meant to go on forever. Can you add parts of the next two rows and columns? Which of these ordered pairs do you think belong to | -- (11, 11), (12, 12), (18, 12), (22, 11), (12, 3), (1540, 308), (7, 35)? Perhaps you have already discovered what relation the picture is meant to describe. [It is a relation you studied in Unit 4.] If so, you know what '|' means.

So, although a picture is sometimes helpful in describing a relation, it does have limitations. A much less ambiguous method is to use brace-notation. Thus, instead of giving a picture of >, we can say that it is

$$\{(x, y) \in R \times R\colon\ y - x \text{ is positive}\},$$

where R is the set of real numbers. Also, instead of the picture of |, we can say that it is

$$\{(x, y) \in I^+ \times I^+\colon\ y \text{ is a factor of } x \text{ with respect to } I^+\},$$

where I^+ is the set of positive integers.

A brace-notation name for a relation enables you to tell whether or not an ordered pair belongs to the relation. The first part of the name--the part on the left of the colon--tells you which ordered pairs are under consideration. The second part of the name is a sentence which can be thought of as a device for selecting from these ordered pairs just those which belong to the relation. [For this reason, such a sentence is often called 'a set selector'.]

In Unit 2 we adopted the convention that unless otherwise specified the domain of a pronumeral is the set of real numbers. So, we can omit '$\epsilon R \times R$' from the first part of a brace-notation name for a relation among the real numbers. Thus, for example, when you see:

$$\{(x, y): \ y \geq 3x + 7\},$$

you are looking at an abbreviation of:

$$\{(x, y) \epsilon R \times R: \ y \geq 3x + 7\}$$

If we wanted to make a great deal of use of this particular relation, we might assign a short, arbitrary name to it, say, 'K'. Then, we could say that $(2, 15) \epsilon K$ and $(5, 20) \notin K$. Indeed, we might even say that $15 \, K \, 2$ or $20 \not K 5$. In most cases, one's interest in a given relation is likely to be too short-lived to make it profitable to give it a short, arbitrary name.

EXERCISES

A. Answer each of the following questions.

1. Suppose M is the relation $\{(x, y): \ x^2 + y^2 = 25\}$. Which of the following sentences are true?

   (a) $(3, 4) \epsilon M$ (b) $(5, 10) \epsilon \widetilde{M}$

   (c) $-3 \, M \, -4$ (d) $(3, 4.08) \epsilon M$

   (e) $(2.9, 3.9) \epsilon M$ (f) $-\sqrt{21} \ \widetilde{M} \ -\sqrt{2}$

2. Suppose B is the relation $\{(p, q) \epsilon I \times I: \ |p| + 1 = |q|\}$, where I is the set of integers. Which of the following sentences are true?

   (a) $(4, 5) \epsilon \cancel{M}$ B (b) $-5 \, B \, 4$

   (c) $(7.8, -8.8) \epsilon B$ (d) $(8, -7) \epsilon B$

   (e) $(-7, 8) \epsilon B$ (f) $-1 \ \widetilde{B} \ 0$

3. Give three ordered pairs which belong to A where

$$A = \{(x, y) \epsilon I \times I: \ 3x + 5y = 9\}.$$

4. Give three ordered pairs which belong to D where

$$D = \{(x, y) \epsilon I \times I: \ 3x + 6y = 7\}.$$

5. Give three ordered pairs of irrational numbers which belong to F, where

$$F = \{(x, y):\ x^2 + y^2 - 7 = 0\}.$$

6. True or false?

(a) $(5, 30) \in \{(x, y):\ y > 5x + 4\}$

(b) $(2, -2) \in \{(x, y):\ x^2 = y^2\}$

(c) $(6, 3) \in \{(p, q),\ p \geq -3:\ q = \sqrt{p + 3}\}$

(d) $(5, 4) \in \{(a, b):\ a + 2b + 7 = 0\}$

(e) $(6, 3) \in \{(p, q),\ q \geq -3:\ p = \sqrt{q + 3}\}$

(f) $(6, 3) \in \{(q, p),\ q \geq -3:\ p = \sqrt{q + 3}\}$

7. For each relation, find an ordered pair which belongs to it.

(a) $\{(x, y):\ y - x = 7\}$

(b) $\{(x, y):\ x - y = 7\}$

(c) $\{(x, y):\ x^2 + y^2 = 100\}$

(d) $\{(x, y):\ y = |x| + 2\}$

(e) $\{(x, y):\ y = 3x^2\}$

(f) $\{(x, y):\ x = 3y^2\}$

(g) $\{(a, b):\ b > 3a - 1\}$

(h) $\{(c, d):\ |c| + |d| > 5\}$

(i) $\{(p, q):\ p + 3q = 9 \text{ and } p \geq 7\}$

(j) $\{(r, s):\ r = 2s \text{ or } r^3 + 3s^2 = 11\}$

(k) $\{(r, s):\ r = s \text{ and } 2r + 5s = 7\}$

(ℓ) $\{(x, y),\ x \in I^+ \text{ and } y \in I^-:\ 3x + 4y = 4\}$, where I^+ is the set of positive integers and I^- is the set of negative integers.

(m) $\{(x, y),\ x \in A \text{ and } y \in B:\ x + 2y = 7\}$, where A is the set of multiples of 3 and B is the set of rational numbers.

(n) $\{(x, y) \in A \times B:\ x + y = 45\}$, where A is the set of multiples of 5 and B is the set of multiples of 6. [Note that '$(x, y) \in A \times B$' is just an abbreviation for '$(x, y),\ x \in A$ and $y \in B$'. The relation in question is a subset of the cartesian product of $A \times B$.]

(o) $\{(x, y) \in M \times W$: x was a married president of the United States and y was his wife$\}$, where M is the set of all men who ever lived and W is the set of all women who ever lived.

(p) $\{(x, y) \in T \times P$: y has played on x$\}$, where T is the set of major league baseball teams and P is the set of major league baseball players, past and present.

* * *

Parts (o) and (p) of Exercise 7 illustrate the fact that the concept of a relation is a very general one. Any set of ordered pairs is a relation, not just sets of ordered pairs of numbers. Note that in the sentence:

(1) y has played on x,

the letters 'y' and 'x' are pronouns just as they are in a sentence like:

(2) $y = 7 + x$

The pronouns in (2) hold places for numerals [that is, for names of numbers], and so we have called such pronouns <u>pronumerals</u>. What do the pronouns in (1) hold places for? If we look back at part (p) of Exercise 7, we see that 'y' holds a place for names of members of the set P, that is, names of major league baseball players. What does 'x' hold a place for? It would be inappropriate to say that the pronouns in (1) are pronumerals, and it is hardly worthwhile to invent special names for such pronouns. It is customary to call all such pronouns, including pronumerals, <u>variables</u>.

<u>A variable is a pronoun</u>. A pronumeral is a kind of variable; actually, it is a numerical variable. The important thing to remember is that a variable is a mark which holds a place in a sentence or in an expression for names of things. The things [numbers, sets, points, people, teams, etc.] whose names are substituted for the variable are called <u>values of the variable</u>; and the set of all values of a variable is called the <u>domain of the variable</u> [or, in some books, <u>the range of the variable</u>].

* * *

B. Draw graphs of these relations. [Use the convention, adopted in Unit 4, of drawing a horizontal line to picture the first component axis, and a vertical line for the second.]

1. $\{(x, y): y = 2x + 4\}$
2. $\{(x, y): x = 2y + 4\}$
3. $\{(x, y): y = 3\}$
4. $\{(x, y): x = 3\}$
5. $\{(a, b): 3a + 2b - 6 = 0\}$
6. $\{(r, s): s \geq |r|\}$
7. $\{(x, y): x = 3 \text{ or } y = 2\}$
8. $\{(x, y): x = 3 \text{ and } y = 2\}$
9. $\{(x, y): y = x^2\}$
10. $\{(x, y): y \geq x^2\}$
11. $\{(x, y): x > 2 \text{ and } y < 3\}$
12. $\{(x, y): y < x < 3\}$
13. $\{(x, y): 2y = x + 5 \text{ and } x \geq 3\}$
14. $\{(x, y): y = 2x - 2 \text{ and } y > 4\}$
15. $\{(x, y): (2y = x + 5 \text{ and } x \geq 3) \text{ or } (y = 2x - 2 \text{ and } y \geq 4)\}$
16. $\{(x, y): 2x + 3y = 26 \text{ or } 5x - 2y = 8\}$
17. $\{(x, y): 2x + 3y = 26 \text{ and } 5x - 2y = 8\}$

[Supplementary exercises on drawing graphs of relations are in Part A on page 5-238. Supplementary exercises which give practice in recognizing graphs of relations are in Part B on pages 5-238 through 5-239.]

INTERSECTIONS AND UNIONS OF RELATIONS

In Exercise 16 of Part B you were required to find the ordered pairs which satisfied at least one of two sentences. So, Exercise 16 required you to graph the union of two relations, that is, to graph

$$\{(x, y): 2x + 3y = 26\} \cup \{(x, y): 5x - 2y = 8\}.$$

On the other hand, in Exercise 17 you were required to find the ordered pairs which satisfied both of two sentences. This is a problem in graphing the intersection of two relations, that is, in graphing

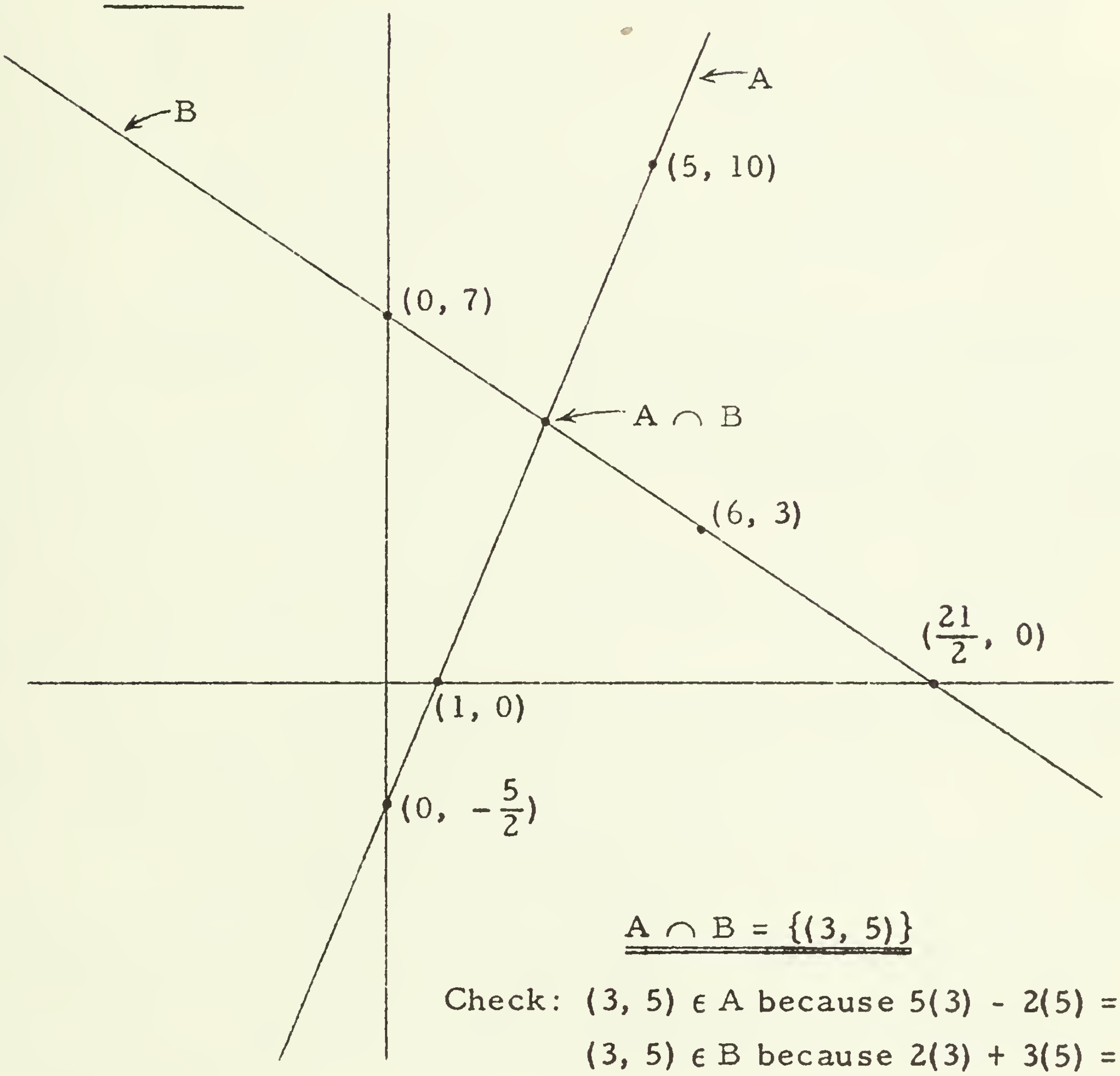
$$\{(x, y): 2x + 3y = 26\} \cap \{(x, y): 5x - 2y = 8\}.$$

EXERCISES

A. Draw graphs and find the ordered pairs in $A \cap B$. Check by substitution.

Sample. $A = \{(x, y): 5x - 2y = 5\}$
$B = \{(x, y): 2x + 3y = 21\}$

Solution.



$A \cap B = \{(3, 5)\}$

Check: $(3, 5) \in A$ because $5(3) - 2(5) = 5$.
$(3, 5) \in B$ because $2(3) + 3(5) = 21$.

The picture above suggests that there is only one ordered pair, (3, 5), which belongs to both A and B. This is the case. So, $A \cap B$ is the set consisting of just the ordered pair (3, 5).

1. $A = \{(x, y): 7x + 2y - 35 = 0\}$
   $B = \{(x, y): 4x - y - 5 = 0\}$

2. $A = \{(x, y): 3x + y = 3\}$
   $B = \{(x, y): 8x + 5y = 1\}$

3. $A = \{(p, q): 3p - 2q + 1 = 0\}$
   $B = \{(r, s): 2r - 3s - 11 = 0\}$

4. $A = \{(x, y): |y| = 3\}$
   $B = \{(x, y): x + 5y = 17\}$

5. $A = \{(x, y): |x| = 4\}$
   $B = \{(x, y): |x| = |y|\}$

6. $A = \{(x, y): y = x^2\}$
   $B = \{(x, y): 2x = y\}$

*

Find the ordered pairs in $A \cup B$.

7. $A = \{(x, y) \in I \times I: x + y < 5, \ x > 0, \text{ and } y > 0\}$
   $B = \{(x, y) \in I \times I: x + y \geq 5, \ x < 5, \text{ and } y < 5\}$

8. $A = \{(x, y) \in I \times I: |x + 2y| = 10 \text{ and } xy \geq 0\}$
   $B = \{(x, y) \in I \times I: |x - 2y| = 10 \text{ and } xy \leq 0\}$

[Supplementary exercises on the graphs of intersections, unions, and complements are in Part C, page 5-240.]

B. Draw graphs of A, B, and $A \cap B$, where

$$A = \{(x, y): y < x + 2\},$$

and $$B = \{(x, y): y > |x - 2|\}.$$

C. 1. Graph the relations A, B, and C, where

$$A = \{(x, y): |x| < 8 \text{ and } |y| = 8\},$$
$$B = \{(x, y): |y| < 8 \text{ and } |x| = 8\},$$

and $$C = \{(x, y): |x| + |y| = 10\}.$$

2. List the members of $(A \cup B) \cap C$.

3. List the members of $(A \cap C) \cup (B \cap C)$.

D. 1. Graph the relations A, B, and C, where

$$A = \{(x, y): y \leq \tfrac{1}{2}x\},$$
$$B = \{(x, y): y \geq -\tfrac{1}{2}x\},$$

and $$C = \{(x, y): x \leq 0 \text{ and } y = 0\}.$$

2. Graph $(A \cap B) \cup C$.

3. Graph $(A \cup C) \cap (B \cup C)$.

5.02 <u>Principles for sets</u>.--Your answers to Exercises 2 and 3 of Part C on page 5-12 may have suggested to you an important generalization about sets. Did you find that, for the sets A, B, and C,

$$(1) \qquad (A \cup B) \cap C = (A \cap C) \cup (B \cap C)?$$

You should have, for this is an instance of the <u>distributive principle for intersecting over unioning</u>, for subsets of the number plane. Does the pattern of sentence (1) remind you of the pattern of instances of one of the principles for <u>numbers</u> you studied in an earlier unit?

If we use 'x', 'y' and 'z' as variables whose domain consists of all subsets [lines, rays, segments, curves, regions, etc.] of the number plane, then (1) is an instance of:

$$(2) \qquad \forall_x \forall_y \forall_z \; (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$$

In general, if the domain of 'x', 'y', and 'z' consists of all subsets of some set S then (2) is the distributive principle for intersecting over unioning, for subsets of S. Let's check this in another case, where S is the set of all whole numbers,

$$A = \{5, 8, 13, 15\},$$

$$B = \{7, 8, 12, 14\},$$

and $$C = \{3, 8, 12, 15\}.$$

[Then A, B, and C are subsets of S.] Is it the case that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)?$$

To answer this question, we first find the members of $(A \cup B) \cap C$. Next, we find those of $(A \cap C) \cup (B \cap C)$. If we find that $(A \cup B) \cap C$ and $(A \cap C) \cup (B \cap C)$ have the same members then we will know that the answer to the question is 'yes'. Now,

$$\begin{aligned} A \cup B &= \{5, 8, 13, 15\} \cup \{7, 8, 12, 14\} \\ &= \{5, 7, 8, 12, 13, 14, 15\}. \end{aligned}$$

So, since

$$\begin{aligned} C &= \{3, 8, 12, 15\}, \\ (A \cup B) \cap C &= \{5, 7, 8, 12, 13, 14, 15\} \cap \{3, 8, 12, 15\} \\ &= \{8, 12, 15\}. \end{aligned}$$

On the other hand,

$$A \cap C = \{5, 8, 13, 15\} \cap \{3, 8, 12, 15\}$$
$$= \{8, 15\},$$

and

$$B \cap C = \{7, 8, 12, 14\} \cap \{3, 8, 12, 15\}$$
$$= \{8, 12\}.$$

So,

$$(A \cap C) \cup (B \cap C) = \{8, 15\} \cup \{8, 12\}$$
$$= \{8, 12, 15\}.$$

Hence, the answer <u>is</u> 'yes'.

Look at Exercises 2 and 3 of Part D on page 5-12. Do they suggest another principle for subsets of the number of plane? What might you call this principle? State it. Do you think that there is a similar principle for sets of whole numbers? Check by using easy sets like A, B, and C of page 5-13.

EXERCISES

<u>A</u>. You learned in your work with numbers that the operation addition, for numbers, is commutative and associative, and that the operation multiplication also has these properties. Do you think that the set operations, intersecting and unioning, for subsets of a given set S, have these properties? For each property and each operation, write a statement which says that the operation has the property. Then, for each of these four statements, pick a few sets [sets of numbers, sets of ordered pairs, sets of students in your class, etc.], form an instance of the statement, and check the instance.

<u>B</u>. The number 0 has two interesting properties which we expressed by principles--the principle for adding 0:

$$\forall_x \; x + 0 = x,$$

and the principle for multiplying by 0:

$$\forall_x \; x \times 0 = 0.$$

Are there analogous principles for sets? State them and check instances.

C. The operations with sets have a property which the operations with numbers do not have. For example, what do you get if you union a set with itself? If you intersect a set with itself? State principles which express the facts that unioning and intersecting have this property. [This property is sometimes called the idempotent property.]

BASIC PRINCIPLES AND THEOREMS

In the preceding exercises you have discovered that some of the principles for operating with subsets of a given set S are analogous to principles for operating with real numbers. Here is a list of ten basic principles for real numbers which you studied in Unit 2.

Commutative principles

$\forall_x \forall_y\ x + y = y + x$

$\forall_x \forall_y\ x \cdot y = y \cdot x$

Associative principles

$\forall_x \forall_y \forall_z\ (x + y) + z = x + (y + z)$

$\forall_x \forall_y \forall_z\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$

Distributive principle

$\forall_x \forall_y \forall_z\ (x + y) \cdot z = x \cdot z + y \cdot z$

Principles for 0 and 1

$\forall_x\ x + 0 = x$

$\forall_x\ x \cdot 1 = x$

Principle of Opposites

$\forall_x\ x + -x = 0$

Principle for Subtraction

$\forall_x \forall_y\ x - y = x + -y$

Principle of Quotients

$\forall_x \forall_{y \neq 0}\ (x \div y) \cdot y = x$

If in seven of these principles for real numbers, you replace '+' by '$\cup$', '$\cdot$' by '$\cap$', '0' by '$\emptyset$', and '1' by 'S', and you adopt the convention that the domain of the variables 'x', 'y', and 'z' is the set of all subsets of S,

you get the following principles for operating with subsets of S:

<u>Commutative principles</u>

$\forall_x \forall_y \; x \cup y = y \cup x$

$\forall_x \forall_y \; x \cap y = y \cap x$

<u>Associative principles</u>

$\forall_x \forall_y \forall_z \; (x \cup y) \cup z = x \cup (y \cup z)$

$\forall_x \forall_y \forall_z \; (x \cap y) \cap z = x \cap (y \cap z)$

<u>Distributive principle</u>

$\forall_x \forall_y \forall_z \; (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$

<u>Principles for $\emptyset$ and S</u>

$\forall_x \; x \cup \emptyset = x$

$\forall_x \; x \cap S = x$

The remaining three principles for real numbers [the po, the ps, and the pq] do not "translate" successfully. However, there are three other basic principles for subsets of S--the other <u>distributive principle</u>:

$$\forall_x \forall_y \forall_z \; (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

and:

<u>Principles for Complements</u>

$\forall_x \; \tilde{x} \cup x = S$

$\forall_x \; \tilde{x} \cap x = \emptyset$

Just as in Unit 2 you derived theorems from the basic principles for real numbers, so from these ten principles one can derive all theorems about unioning, intersecting, and complementing subsets of S. To see how this could be done, let's find test-patterns for <u>the uniqueness principle for intersecting</u>:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x \cap z = y \cap z,$$

and <u>the principle for intersecting with $\emptyset$</u>:

$$\forall_x \; x \cap \emptyset = \emptyset$$

Here is a test-pattern for the uniqueness principle. [You may want to compare it with the test-pattern near the bottom of page 2-64.]

Suppose that	$x = y$.
Since	$x \cap z = x \cap z$, $[\forall_a\ a = a]$
it follows that	$x \cap z = y \cap z$.
Hence,	if $x = y$ then $x \cap z = y \cap z$.

Here is a test-pattern for the principle for intersecting with $\emptyset$. [Since there is no operation for subsets which is analogous to the oppositing operation for real numbers, this test-pattern is quite different from the one on page 2-66 for the principle for multiplying by 0.]

$\tilde{x} \cup \emptyset = \tilde{x}$	$[\forall_x\ x \cup \emptyset = x]$
$(\tilde{x} \cup \emptyset) \cap x = \tilde{x} \cap x$	$[\forall_x\forall_y\forall_z$ if $x = y$ then $x \cap z = y \cap z]$
$(\tilde{x} \cap x) \cup (\emptyset \cap x) = \tilde{x} \cap x$	$[\forall_x\forall_y\forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)]$
$\emptyset \cup (\emptyset \cap x) = \emptyset$	$[\forall_x\ \tilde{x} \cap x = \emptyset]$
$(\emptyset \cap x) \cup \emptyset = \emptyset$	$[\forall_x\forall_y\ x \cup y = y \cup x]$
$\emptyset \cap x = \emptyset$	$[\forall_x\ x \cup \emptyset = x]$
$x \cap \emptyset = \emptyset$	$[\forall_x\forall_y\ x \cap y = y \cap x]$

* * *

<u>D</u>. 1. Rewrite the test-pattern for the principle for intersecting with $\emptyset$, but write '$\cap$' instead of '$\cup$', '$\cup$' instead of '$\cap$', and 'S' instead of '$\emptyset$'. [Your last line should be '$x \cup S = S$'.] Are the bracketed expressions on the right of your new pattern principles for subsets of S?

2. Rewrite the ten basic principles for subsets of S, but write '$\cap$' for '$\cup$', '$\cup$' for '$\cap$', 'S' for '$\emptyset$' and '$\emptyset$' for 'S'. What do you notice?

3. Suppose you have derived a theorem about subsets of S from the ten basic principles. How can you find another theorem?

4. Here is a test-pattern for the theorem '$\forall_x\ x = x \cup x$', the idempotence theorem for unioning.

$x = x \cup \emptyset$	$[\forall_x\ x \cup \emptyset = x]$
$= \emptyset \cup x$	$[\forall_x \forall_y\ x \cup y = y \cup x]$
$= (\tilde{x} \cap x) \cup x$	$[\forall_x\ \tilde{x} \cap x = \emptyset]$
$= (\tilde{x} \cup x) \cap (x \cup x)$	$[\forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)]$
$= S \cap (x \cup x)$	$[\forall_x\ \tilde{x} \cup x = S]$
$= (x \cup x) \cap S$	$[\forall_x \forall_y\ x \cap y = y \cap x]$
$= x \cup x$	$[\forall_x\ x \cap S = x]$

Write a test-pattern for '$\forall_x\ x = x \cap x$', the idempotence theorem for intersecting.

E. One of the important theorems you proved about the oppositing operation for real numbers was the 0-sum theorem:

$$\forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y$$

For subsets of S, we have the theorem:

$$\forall_x \forall_y \text{ if } x \cup y = \emptyset \text{ then } y = \emptyset$$

Prove this theorem by completing the test-pattern which is started below.

Suppose that $x \cup y = \emptyset$.

Then $y = y \cup (x \cup y)$ $[\forall_x\ x \cup \emptyset = x]$

$= \_\_\_\_\_\_\_\_\_\_$ $[\forall_x \forall_y\ x \cup y = y \cup x]$

.
.
.

Note: The rest of this section contains optional material on subsets. Mastery of this material is not necessary for understanding of later work in Unit 5. However, some of the theorems proved here will be helpful, so, we list them in a summary on page 5-22. Be sure to familiarize yourself with this summary even if you don't study the optional material.

☆ MORE THEOREMS ABOUT SUBSETS

The theorem of Part E, on page 5-18, about subsets of S shows that there is no satisfactory analogue, for subsets, of the oppositing operation for real numbers [Why?]. However, there is a theorem about complementing which is somewhat analogous to the 0-sum theorem. We shall call it <u>the complement theorem</u>:

$$\forall_x \forall_y \text{ if } x \cup y = S \text{ and } x \cap y = \emptyset \text{ then } \tilde{x} = y$$

The complement theorem can be derived from the ten basic principles for subsets, but we shall not do so here. You can easily convince yourself by an example that the complement theorem is true. Take $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 2, 3\}$, and try to find a set B such that $A \cup B = S$ and $A \cap B = \emptyset$. What must B be? Is $B = \tilde{A}$?

You can use the complement theorem to prove several useful theorems about complements. For example, from the principles for $\emptyset$ and S on page 5-16, it follows that $S \cup \emptyset = S$ and $\emptyset \cap S = \emptyset$. Hence [since $\emptyset \cap S = S \cap \emptyset$], it follows from the complement theorem that $\tilde{S} = \emptyset$. [Since $S \cup \emptyset = \emptyset \cup S$, it follows in the same manner that $\tilde{\emptyset} = S$.]

As another simple example, notice that the principles for complements on page 5-16 and the complement theorem tell you at once that

$$\forall_x \tilde{\tilde{x}} = x.$$

A more interesting application of the complement theorem is to prove:

<u>DeMorgan's Laws</u>

$$\forall_x \forall_y \widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$$

$$\forall_x \forall_y \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$$

You may have discovered these when solving exercises in Part F on page 5-I. The first can be interpreted as saying that the members of S which don't belong to either x or y are just those members of S which belong neither to x nor to y. What does the second tell you about the members of S which don't belong both to x and to y?

Here is a test-pattern for the first of DeMorgan's Laws:

$$
\begin{array}{ll}
(x \cup y) \cap (\tilde{x} \cap \tilde{y}) & \\
 & \Big\rangle \forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z) \\
= [x \cap (\tilde{x} \cap \tilde{y})] \cup [y \cap (\tilde{x} \cap \tilde{y})] & \\
 & \Big\rangle \forall_x \forall_y\ x \cap y = y \cap x \\
= [x \cap (\tilde{x} \cap \tilde{y})] \cup [y \cap (\tilde{y} \cap \tilde{x})] & \\
 & \Big\rangle \forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z) \\
= [(x \cap \tilde{x}) \cap \tilde{y}] \cup [(y \cap \tilde{y}) \cap \tilde{x}] & \\
 & \Big\rangle \forall_x \forall_y\ x \cap y = y \cap x \\
= [\tilde{y} \cap (\tilde{x} \cap x)] \cup [\tilde{x} \cap (\tilde{y} \cap y)] & \\
 & \Big\rangle \forall_x\ \tilde{x} \cap x = \emptyset \\
= [\tilde{y} \cap \emptyset] \cup [\tilde{x} \cap \emptyset] & \\
 & \Big\rangle \forall_x\ x \cap \emptyset = \emptyset \\
= \emptyset \cup \emptyset & \\
 & \Big\rangle \forall_x\ x \cup x = x \\
= \emptyset &
\end{array}
$$

[Having gotten this far, we know that '$\forall_x \forall_y\ (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset$' is a theorem. So, we know that '$\forall_x \forall_y\ (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S$' is also a theorem [Why?]. We can use this in the remainder of this proof.]

$$
\begin{array}{lll}
(x \cup y) \cup (\tilde{x} \cap \tilde{y}) & = (\tilde{x} \cap \tilde{y}) \cup (x \cup y) & [\forall_x \forall_y\ x \cup y = y \cup x] \\
 & = (\tilde{x} \cap \tilde{y}) \cup (\tilde{\tilde{x}} \cup \tilde{\tilde{y}}) & [\forall_x\ \tilde{\tilde{x}} = x] \\
 & = S & [\forall_x \forall_y\ (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S]
\end{array}
$$

We have shown that

$$(x \cup y) \cup (\tilde{x} \cap \tilde{y}) = S \text{ and } (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset.$$

So, by the complement theorem, $\widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$.

EXERCISES

A. Write a test-pattern for each of the given generalizations.

1. $\forall_x\ \tilde{\tilde{x}} = x$
2. $\forall_x \forall_y\ \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$

* * *

So far we have discussed only the operations $\cup$, $\cap$ and $\tilde{}$. In addition to these, there is the relation $\subseteq$ to be considered. For this

we need one more basic principle:

Inclusion principle

$$\forall_x \forall_y \ [y \subseteq x \text{ if and only if } x \cap y = y]$$

One consequence of the inclusion principle is, of course:

$$\forall_x \forall_y \text{ if } x \cap y = y \text{ then } y \subseteq x$$

Using this it is easy to show that, for example, '$\forall_x\, x \subseteq x$' is a theorem. Here is a test-pattern:

	$x \cap x = x$	$[\forall_x\, x \cap x = x]$
But,	if $x \cap x = x$ then $x \subseteq x$.	$[\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$
Hence,	$x \subseteq x$.	

Another consequence of the inclusion principle is:

$$\forall_x \forall_y \text{ if } y \subseteq x \text{ then } x \cap y = y$$

Using this we can show, for example, that '$\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq x$ then $x = y$' is a theorem. Here is a test-pattern:

Suppose that $x \subseteq y$ and $y \subseteq x$.

Then,	$y \cap x = x$ and $x \cap y = y$.	$[\forall_x \forall_y$ if $y \subseteq x$ then $x \cap y = y]$
So,	$x \cap y = x$ and $x \cap y = y$.	$[\forall_x \forall_y\, x \cap y = y \cap x]$
Hence,	$x = y$.	

Consequently, if $x \subseteq y$ and $y \subseteq x$ then $x = y$.

* * *

<u>B</u>. Write test-patterns for each of the given generalizations.

1. $\forall_x\, \emptyset \subseteq x$ 2. $\forall_x\, x \subseteq S$ 3. $\forall_x \forall_y\, x \cap y \subseteq x$

4. $\forall_x \forall_y\, x \subseteq x \cup y$ [<u>Hint</u>. Here is a first step in a test-pattern for Exercise 4:

$$(x \cup y) \cap x = (x \cup y) \cap (x \cup \emptyset) \ [\forall_x\, x \cup \emptyset = x]$$]

5. $\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$

SUMMARY

BASIC PRINCIPLES

Commutative principles

$\forall_x \forall_y\ x \cup y = y \cup x$

$\forall_x \forall_y\ x \cap y = y \cap x$

Associative principles

$\forall_x \forall_y \forall_z\ (x \cup y) \cup z = x \cup (y \cup z)$

$\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)$

Distributive principles

$\forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$

$\forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$

Principles for $\emptyset$ and S

$\forall_x\ x \cup \emptyset = x$

$\forall_x\ x \cap S = x$

Principles for complements

$\forall_x\ \tilde{x} \cup x = S$

$\forall_x\ \tilde{x} \cap x = \emptyset$

Inclusion principle

$\forall_x \forall_y\ [y \subseteq x$ if and only if $x \cap y = y]$

THEOREMS

1. (a) $\forall_x \forall_y \forall_z$ if $x = y$ then $x \cup z = y \cup z$

   (b) $\forall_x \forall_y \forall_z$ if $x = y$ then $x \cap z = y \cap z$

2. (a) $\forall_x\ x \cup S = S$ (b) $\forall_x\ x \cap \emptyset = \emptyset$

3. (a) $\forall_x\ x = x \cup x$ (b) $\forall_x\ x = x \cap x$

4. (a) $\forall_x \forall_y$ if $x \cup y = \emptyset$ then $y = \emptyset$ (b) $\forall_x \forall_y$ if $x \cap y = S$ then $y = S$

5. $\forall_x \forall_y$ if $x \cup y = S$ and $x \cap y = \emptyset$ then $x = y$

6. (a) $\tilde{S} = \emptyset$ (b) $\tilde{\emptyset} = S$

7. $\forall_x\ \tilde{\tilde{x}} = x$

8. (a) $\forall_x \forall_y\ \widetilde{x \cup y} = \tilde{x} \cap \tilde{y}$ (b) $\forall_x \forall_y\ \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$

9. (a) $\forall_x\ \emptyset \subseteq x$ (b) $\forall_x\ x \subseteq S$

10. (a) $\forall_x \forall_y\ x \subseteq x \cup y$ (b) $\forall_x \forall_y\ x \cap y \subseteq x$

11. $\forall_x \forall_y$ $x \subseteq y$ if and only if $x \cup y = y$

12. $\forall_x\ x \subseteq x$

13. $\forall_x \forall_y$ if $x \subseteq y$ and $y \subseteq x$ then $x = y$

14. $\forall_x \forall_y \forall_z$ if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$

* * *

The relation of membership in a set [denoted by 'ϵ'] plays no role in the preceding discussion of subsets. For a different approach to the subject, one which makes much use of ϵ, we can start from the following basic principles. [In stating these principles, we use 'x' and 'y', as above, for variables whose domain consists of the subsets of S, and 'e' as a variable whose domain consists of the members of S. So, the domain of 'e' is S.]

P1. $\forall_x \forall_y \forall_e$ [$e \in x \cup y$ if and only if ($e \in x$ or $e \in y$)]

P2. $\forall_x \forall_y \forall_e$ [$e \in x \cap y$ if and only if ($e \in x$ and $e \in y$)]

P3. $\forall_x \forall_e$ [$e \in \tilde{x}$ if and only if $e \notin x$]

P4. $\forall_e\ e \notin \emptyset$

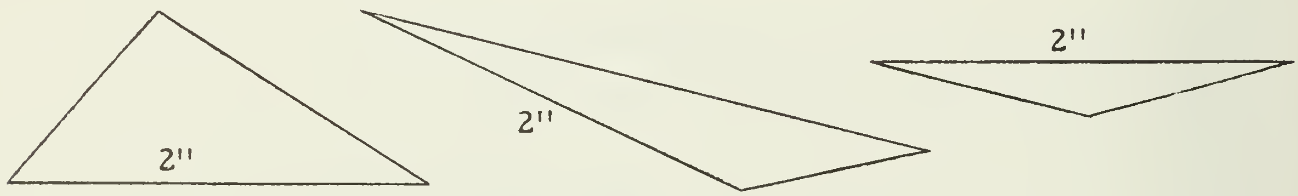
P5. $\forall_e\ e \in S$

P6. $\forall_x \forall_y$ [$y \subseteq x$ if and only if $\forall_e$ if $e \in y$ then $e \in x$]

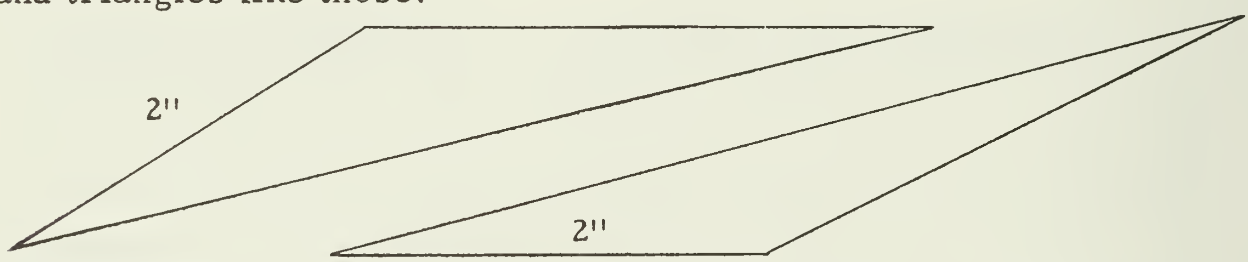
P7. $\forall_x \forall_y$ if $y \subseteq x$ and $x \subseteq y$ then $x = y$

One can use these seven new basic principles to prove the eleven basic principles stated at the beginning of this SUMMARY. So, all the theorems which can be derived from the original basic principles can also be derived from these new basic principles.

5.03 <u>Relations and geometric figures</u>. --Suppose that each person in your class draws a triangle, and that each of these triangles has one side which is 2 inches long. What are the inch-measures of the other two sides? This sounds like a silly question because, as you can readily imagine, there are lots of possibilities for the inch-measures of two sides of a triangle whose third side has inch-measure 2. You could have triangles like these:



and triangles like these:



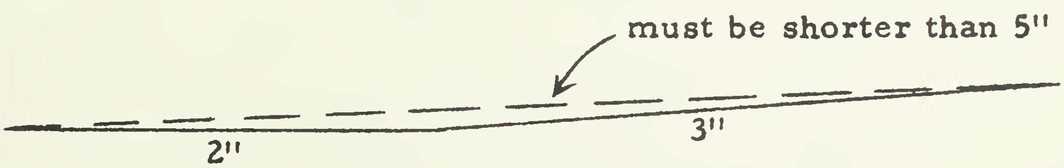
and many more. We can't predict the inch-measures of the other two sides of such triangles. But, is there anything we <u>can</u> say about these inch-measures? For example, if you knew that the inch-measure of one of the two sides was 3, what could you say about the inch-measure of the other side? Could it be 4? 1? 0.5? 5? 7?

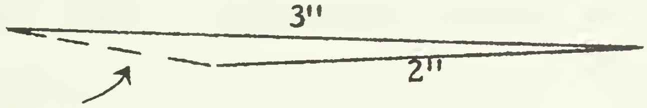
As you are probably beginning to suspect, there is some relation of the inch-measure of one side to the inch-measure of a second side of a triangle whose third side is 2 inches long. We shall now find out just what this relation is. As mentioned earlier, a relation is a set of ordered pairs. The relation in question is a set of ordered pairs of inch-measures of sides of triangles each having a 2-inch third side. Since the inch-measure of a side of a triangle can be any number of arithmetic greater than 0, this is a relation among the nonzero numbers of arithmetic. Our problem, now, is to find out which ordered pairs of nonzero numbers of arithmetic have components which can be the inch-measures of two sides of a triangle whose third side has inch-measure 2.

One way to approach this problem is to try to test every ordered

pair. That is, pick an ordered pair, say, (4, 7), and see if you can draw a triangle whose sides have inch-measures 4, 7, and 2, going counterclockwise. If you can, (4, 7) belongs to the relation. If you can't, you know that (4, 7) doesn't belong. Now, obviously, you can't test all of the ordered pairs since there are infinitely many of them. Also, some of them have such large components that it wouldn't be practical to test them by making a drawing. So, we shall have to be clever about this, use our imagination a great deal, search for patterns, and perhaps make just a few drawings to test some crucial hypotheses.

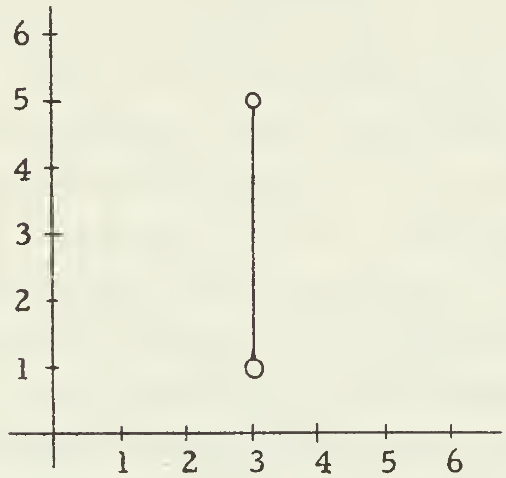
Let's start by deciding to describe the relation by making a graph of it. Since the relation is a subset of the cartesian square of the set of nonzero numbers of arithmetic, we shall graph the relation on a picture of the first quadrant of the number plane. [Recall that the positive real numbers behave just like the nonzero numbers of arithmetic. So, instead of an ordered pair of nonzero numbers of arithmetic we can think of the corresponding ordered pair of positive real numbers.] Now, what points shall we plot? Consider ordered pairs with first component 3. That is, consider triangles whose first and third sides have inch-measures 3 and 2. What are the possible inch-measures for the other side? Here are pictures showing two rather extreme positions for the 3-inch side.





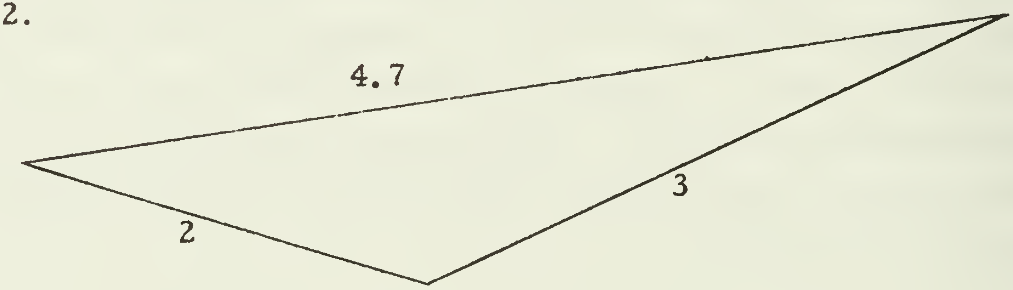
It seems clear that the inch-measure of the second side cannot exceed 5 or be less than 1. In fact, it can't be as large as 5 or as small as 1. [Why?] The second side must have an inch-measure between 1 and 5, and it seems reasonable that there are triangles with sides whose inch-measures are 2, 3, and any number between 1 and 5. So, a

subset of the relation we are looking for is the interval whose end points are (3, 1) and (3, 5). Let's graph this interval.

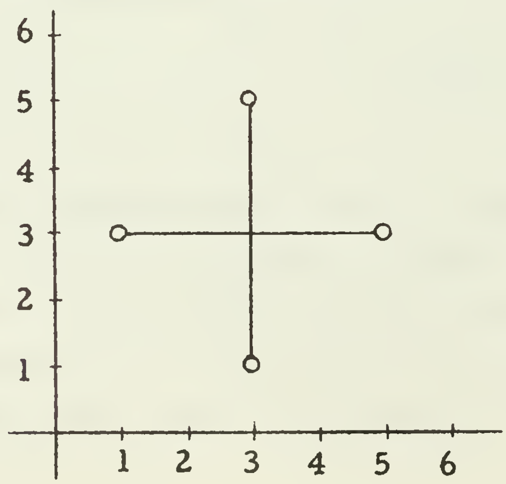


[The hollow dots mean that the corresponding points are not in the relation.]

Now, let's use a bit more imagination to get additional points of the relation. Notice that (3, 4.7) belongs to the relation. Does (4.7, 3) also belong? Here is a picture of a triangle whose sides measure 3, 4.7, and 2.



Can you draw a triangle whose sides measure 4.7, 3, and 2? Do so. [You can hold an edge of a mirror next to the picture of the triangle shown above and look at its reflection. The reflection is a triangle whose sides measure 4.7, 3, and 2.] So, for each point of the interval graphed above, the point with components in the opposite order also belong to the relation. This gives us more points to graph.



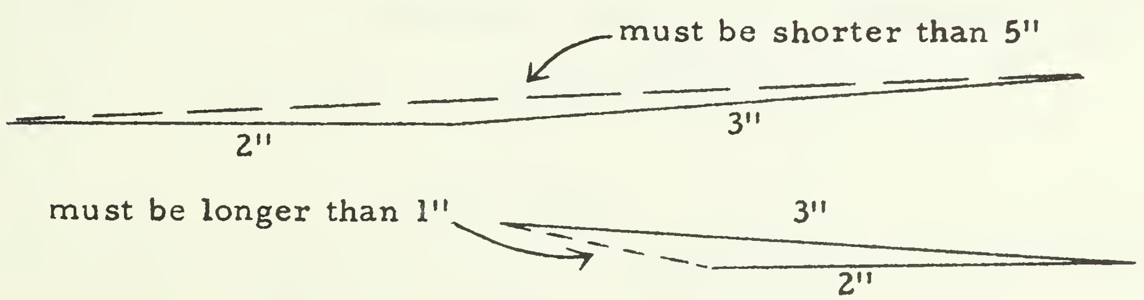
We have discovered an important property of this relation, the property of being symmetric. This is a useful discovery because it helps us find more points in the relation. For example, suppose we know that (101, 100) belongs to the relation. From this, and the fact that the relation is symmetric, we can deduce that what other point belongs to the relation?

There are some points in the relation for which the property of symmetry gives us no additional information. What points are these? What special name do you have for triangles whose side-measures are the components of such points? Is (75, 75) such a point? How about (5, 5)? (2, 2)? Does (0.5, 0.5) belong to the relation? What about (1, 1)? Add more points to the picture on the preceding page by graphing those points in the relation which have equal components. [What special name do we give to triangles with a side of measure 2 whose other side-measures are the components of (2, 2)?]

[In the next section you will learn that a relation is said to be reflexive if and only if it contains all those ordered pairs with equal components which belong to the smallest cartesian square of which the relation is a subset. Although the relation we are now investigating is symmetric, it is not reflexive. Tell why.]

You should now complete the graph of the relation.

Although you have a graph of this relation, it may also be helpful to have a brace-notation name for it. To get such a name, we need a sentence containing two variables, say 'x' and 'y', whose solution set is the relation. Recall the situation discussed on page 5-25. There you discovered that if y is the inch-measure of one side of a triangle

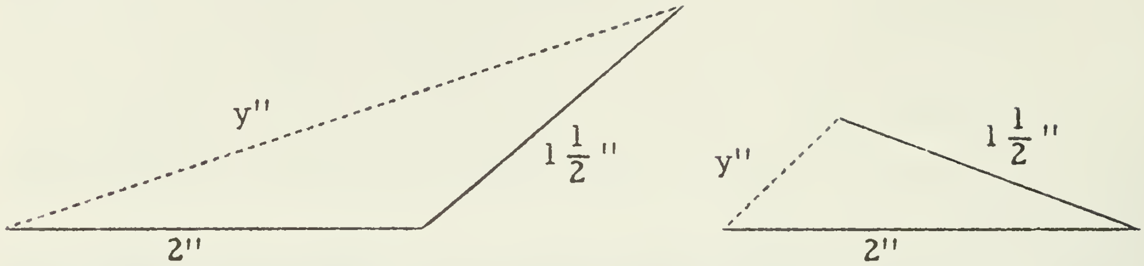


whose other sides are 2 inches and 3 inches long, then

$$y < 3 + 2 \quad \text{and} \quad y > 3 - 2.$$

It also seemed reasonable that for any number which satisfied these inequations, there is such a triangle.

Now, consider this situation:



Here we see that a number can be the inch-measure of a side of a triangle whose other sides are 2 inches and $1\frac{1}{2}$ inches long if and only if it satisfies the sentence:

$$y < 1\frac{1}{2} + 2 \quad \text{and} \quad y > 2 - 1\frac{1}{2}$$

So, in general, y and x can be inch-measures of two sides of a triangle whose third side is 2 inches long if and only if

$$(*) \quad y < x + 2 \quad \text{and} \quad \begin{cases} \text{either } (x \geq 2 \text{ and } y > x - 2) \\ \text{or } (x < 2 \text{ and } y > 2 - x). \end{cases}$$

We can simplify this considerably by noting that

$$\text{if } x \geq 2 \text{ then } x - 2 = {}^{+}|x - 2|$$

and

$$\text{if } x < 2 \text{ then } 2 - x = {}^{+}|x - 2|.$$

Since either $x \geq 2$ or $x < 2$, (*) boils down to:

$$y < x + 2 \text{ and } y > {}^{+}|x - 2|,$$

or, for short:

$${}^{+}|x - 2| < y < x + 2$$

Thus, the relation we have been discussing is

$$\{(x, y): \ |x - 2| < y < x + 2\}.$$

EXERCISES

A. 1. Make a quick sketch of a graph of the relation of the inch-measure of one side of a triangle to the inch-measure of another if the third side is 5 inches long.

2. Write a brace-notation name for this relation.

3. The inch-measures of the sides of some triangle are

   (a) 6, m, and 2 if and only if $4 < m < 8$,

   (b) 7, n, and 5 if and only if ________,

   (c) 16, 2, and q if and only if ________,

   (d) 20, 60, and r if and only if ________,

   (e) 11, a, and b if and only if ________,

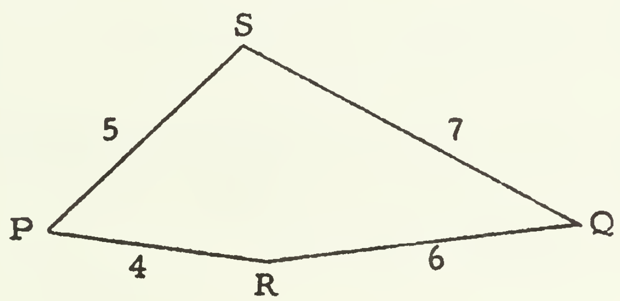
   (f) x, y, and z if and only if ________.

☆ B. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of an isosceles triangle whose third side is 3 inches long.

☆ C. 1. Draw a graph of the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 4 inches long and whose inch-perimeter does not exceed 12.

2. How many such differently-shaped triangles are there whose sides have whole numbers for inch-measures?

D. Four sticks are fastened at their ends to form a quadrilateral as shown.



The quadrilateral is not fixed. That is, it is possible to change its shape by moving, for example, P towards Q, without bending the sticks. How far apart can you move P from Q? How close together can you bring P and Q?

E. For each ordered triple listed below, tell whether its components can be inch-measures of the sides of a triangle.

1. (2, 5, 8)	2. (9, 11, 20)	3. (8, 8, 16)
4. (16, 16, 8)	5. (3, 3, 3)	6. (3, 4, 5)
7. (1, 2, 3)	8. (10, 20, 29)	9. (6, 12, 4)

F. Here is another question dealing with a relation between measures of parts of a geometric figure.

> What is the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°?

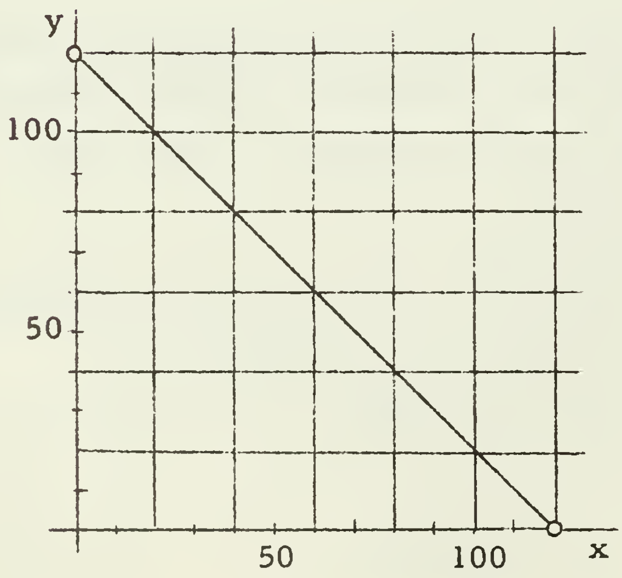
1. Draw at least six triangles each with an angle of 50°, find the degree-measure of each of the other two angles, and plot the corresponding ordered pairs.

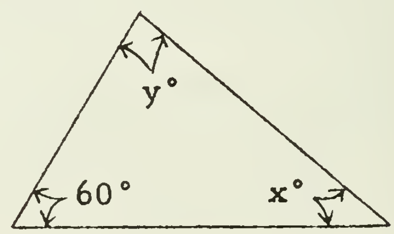
2. Are the six ordered pairs you plotted in Exercise 1 enough to show you the pattern for the rest of the relation? If not, draw more triangles with a 50° angle, measure each of the other two angles, and plot ordered pairs until you do see the pattern of the relation. Is the relation symmetric?

3. Complete the following to get a brace-notation name for the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°.

$$\{(x, y): \ x > 0, \ y > 0, \ \text{and } x + y = \qquad \}.$$

G. Here is a graph of the relation between the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 60°.





1. Write a brace-notation name for this relation.

2. Use the graph or the name of the relation to complete the following table. [$\angle A$, $\angle B$, and $\angle C$ are the three angles of a triangle.]

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
$\angle A$	60°	60°	60°			60°	60°	60°
$\angle B$	20°	100°		60°	60°	39°	40°	120°
$\angle C$			30°	110°	60°			

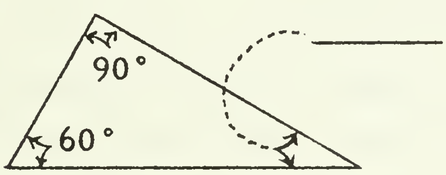
3. Sketch on the chart on page 5-30 a graph of the relation of the degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is one of 80°. Write its name.

4. (a) Which points of the 80°-relation and of the 60°-relation correspond with isosceles triangles?
   (b) Which points of the 60°-relation and of the 80°-relation correspond with equilateral triangles?

5. (a) Which points of the 80°-relation correspond with triangles in which the degree-measure of one angle is twice the degree-measure of another? What are the degree-measures of the three angles of such triangles?
   (b) Repeat for the 60°-relation.

6. Predict the missing angle-measures, and use a protractor to check your predictions.

(a)



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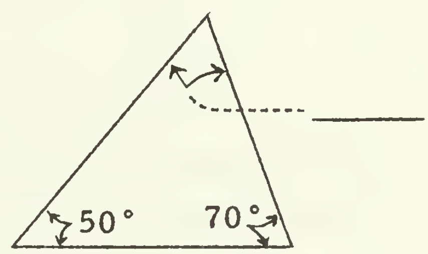

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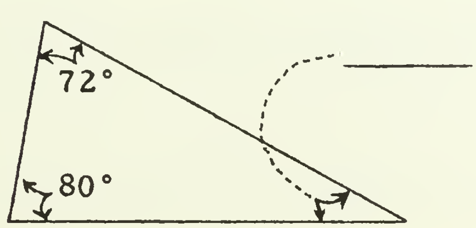

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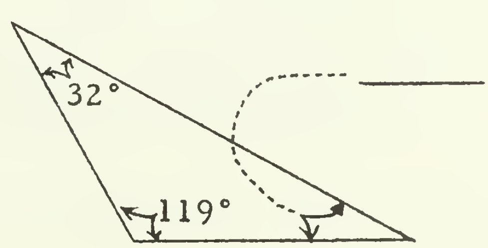
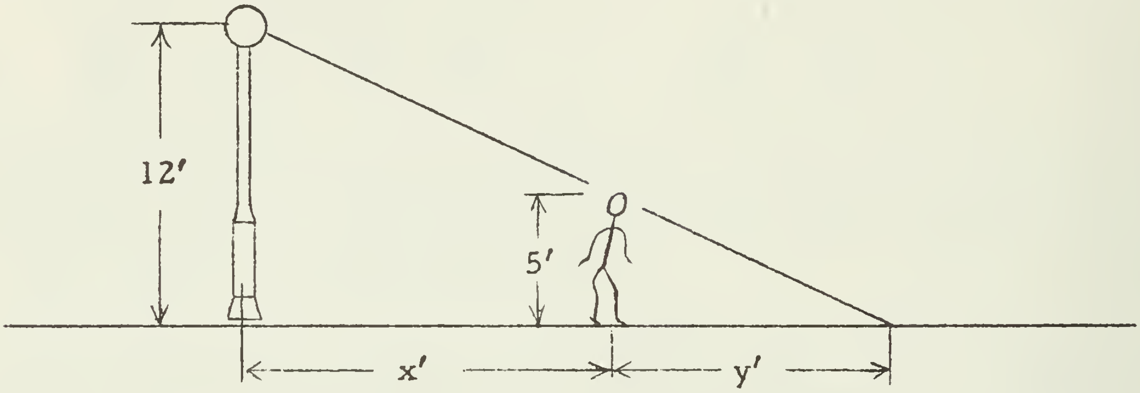


☆H. You may have had the experience of walking away from a lamppost and watching your shadow lengthen as you get farther from the post.

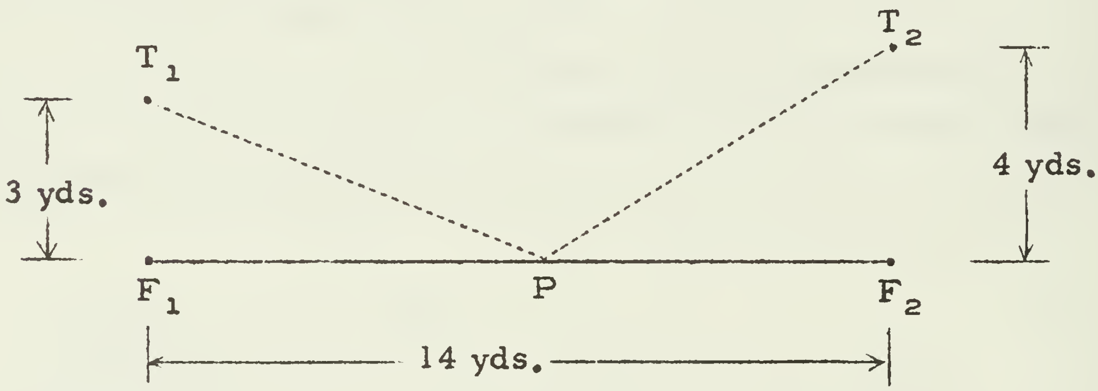


1. Make a graph of the relation of the foot-measure of the shadow of a 5-foot walker to his foot-distance from the 12-foot lamppost.

2. How far is the walker from the post when his shadow is as long as he is tall?

3. How far is the walker from the post when his shadow is twice as long as he is tall?

☆I. Here is a map showing a fence and two trees. If you walk from tree



T_1 to a point P on the fence and from there to tree T_2, the distance walked is the sum of the distances from T_1 to P and from P to T_2.

1. Make a graph of the relation of the distance walked to the distance between F_1 and P.

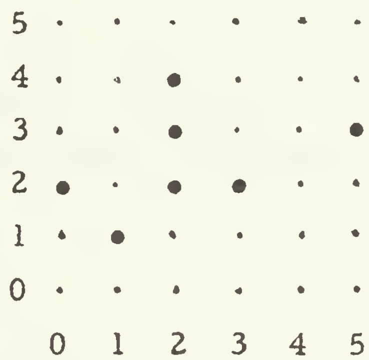
2. Use the graph to tell that location of P for which the distance walked is the smallest.

[Supplementary exercises are in Part D, pages 5-241 through 5-242.]

EXPLORATION EXERCISES

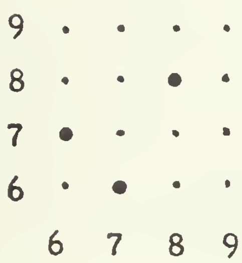
A. Here are graphs or brace-notation names of several relations. For each relation, (a) describe the set of things which are first components, and (b) the set of things which are second components of the members of the relation.

Sample.

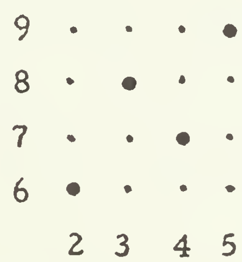


Solution. (a) first components: $\{0, 1, 2, 3, 5\}$
(b) second components: $\{1, 2, 3, 4\}$

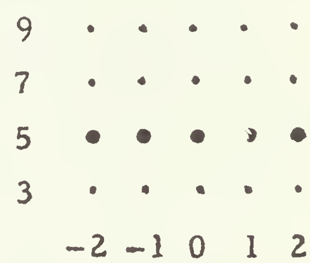
1.



2.



3.



4. $\{(x, y): 1 \leq y \leq 2\}$
5. $\{(x, y): x = 4\}$
6. $\{(x, y): y = x^2\}$
7. $\{(x, y): |x| < 2 \text{ and } |y| < 4\}$
8. $\{(x, y): y = 1 - x \text{ and } x \geq 2\}$
9. $\{(a, b): a^2 + b^2 = 25\}$
10. $\{(a, b): a^2 + b^2 = 36\}$
11. $\{(a, b): a^2 + b^2 = 36 \text{ and } b \geq 0\}$
12. $\{(a, b): a^2 + b^2 = 36 \text{ and } a \geq 0\}$
13. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab > 0\}$
14. $\{(a, b): a^2 + b^2 = 36 \text{ and } ab \geq 0\}$

15. $\{(x, y) \in U \times C$: y is a city in $x\}$, where U is the set of all states in the United States and C is the set of all cities in the United States.

16. $\{(x, y) \in U \times C$: y is the capital of $x\}$

17. $\{(x, y) \in P \times P$: y is an uncle of $x\}$, where P is the set of all people.

B. For each of the relations given in Part A, describe the set of things which are either first or second components of the members of the relation.

C. True or false?

1. There is a real number x such that $3 + 2x = 15$.
2. There is a real number y such that $y^2 + 4^2 = 5^2$.
3. There is a real number y such that $y^2 + 6^2 = 5^2$.
4. There is a real number y such that $y + 1 = y + 2$.
5. There is a real number y such that $y + 1 = 1 + y$.
6. There is an integer q such that $22 - 7 = 5q$.
7. There is an integer q such that $22 - 8 = 5q$.
8. There is a real number y such that, for each real number x, $xy = 0$.
9. There is a real number y such that, for each real number $x \neq 0$, $xy = 1$.
10. For each real number $x \neq 0$, there is a real number y such that $xy = 1$.
11. For each real number x, there is a real number y such that $xy = 0$.
12. There is a real number y such that, for each real number x, $x + y = 0$.
13. For each real number x, there is a real number y such that $x + y = 0$.

5.04 Properties of relations. --In the preceding section we mentioned two of the properties [symmetry and reflexiveness] which relations can have. We want to give precise descriptions of these properties, and in order to do so, we need more terminology.

DOMAIN AND RANGE OF A RELATION

We have said that a relation R among the elements of a set S is a subset of the cartesian square $S \times S$. It may not be the case that each member of S is the first component of a member of R, nor that each member of S is the second component of a member of R. For example, consider the relation U of being-an-uncle-of among the members of the set P of all people. There are people who are not uncles, that is, there are members of P who are not second components of members of U. [Give an example.] Also, there are people who are neither nephews nor nieces. [Is it possible that a nephew not be the first component of a member of U?]

Given a relation R among the members of a set S, it is often convenient to talk about the set of those members of S which are first components of members of R. This set is called the domain of R. [What is the domain of U?] The set of members of S which are second components of members of R is called the range of R. [What is the range of U?] Concisely,

(1) $\mathfrak{D}_R = \{x \in S\text{: there is a } y \in S \text{ such that } (x, y) \in R\}$,

(2) $\mathfrak{R}_R = \{x \in S\text{: there is a } y \in S \text{ such that } (y, x) \in R\}$.

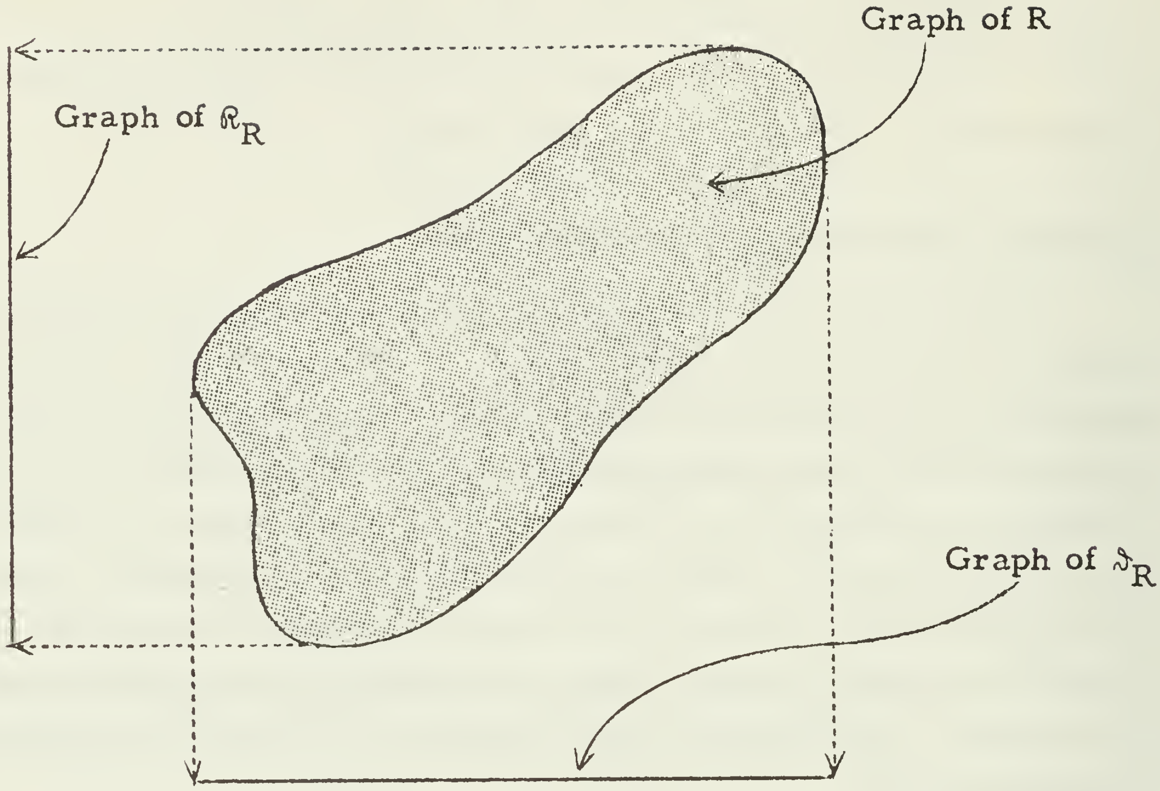
Notice that 'there is a $y \in S$ such that' is a short way of saying 'there is at least one $y \in S$ such that', and it is customary to abbreviate both of these expressions by:

$$\exists_{y \in S}$$

[The symbol '$\exists$' is called an existential quantifier. You are already acquainted with the universal quantifier '$\forall$'.]

So, $\mathfrak{D}_R = \{x \in S\colon \exists_{y \in S} (x, y) \in R\}$,

and, $\mathfrak{R}_R = \{x \in S\colon \exists_{y \in S} (y, x) \in R\}$.



In the case of the relation U,

$$\mathfrak{D}_U = \{x \in P \colon \exists_{y \in P} (x, y) \in U\},$$

and

$$\mathfrak{R}_U = \{x \in P \colon \exists_{y \in P} (y, x) \in U\}.$$

The domain of U is that subset of P which consists of all people who have uncles. The range of U is that subset of P which consists of all people who are uncles. How do we tell if a given element of P, say, Mr. Adams, belongs to $\mathfrak{R}_U$? By definition, Mr. Adams $\in \mathfrak{R}_U$ if and only if the sentence:

$$\exists_{y \in P} (y, \text{Mr. Adams}) \in U$$

is true. And, this sentence is true if and only if the sentence:

$$(y, \text{Mr. Adams}) \in U$$

has at least one solution. [It may be the case that this last sentence has more than one solution. This additional information may be of interest, but it is irrelevant to the question of whether Mr. Adams belongs to $\mathfrak{R}_U$.] What sentence must have at least one solution if Bill Smith is in $\mathfrak{D}_U$? If Mr. Adams is in $\mathfrak{D}_U$?

FIELD OF A RELATION

We need one additional notion, that of <u>the field of R</u>.

$$\mathfrak{F}_R = \mathfrak{D}_R \cup \mathfrak{R}_R .$$

Roughly speaking, $\mathfrak{F}_R$ consists of the members of S which "get into the act". That is, $\mathfrak{F}_R$ is the smallest subset of S whose cartesian square contains all the members of R. So, for example, $\mathfrak{F}_U$ is the set of all people who have or are uncles.

EXERCISES

<u>A</u>. For each relation described below, give its domain, range, and field.

<u>Sample 1</u>. The relation A where

$$A = \{(0, 6), (3, 5), (3, 7), (4, 2), (5, 2), (6, 0)\}.$$

<u>Solution</u>. $\mathfrak{D}_A = \{0, 3, 4, 5, 6\}$

$\mathfrak{R}_A = \{0, 2, 5, 6, 7\}$

$\mathfrak{F}_A = \{0, 2, 3, 4, 5, 6, 7\}$

<u>Sample 2</u>. The relation B where $B = \{(x, y) : y^2 = x + 1\}$.

<u>Solution</u>. To find the domain one needs to use the fact that a real number is a square if and only if it is nonnegative. So,

$$\mathfrak{D}_B = \{x : x \geq -1\}.$$

To find the range one needs to use the fact that each real number is the sum of a real number and 1. It follows that

$\mathfrak{R}_B$ is the set of real numbers.

Hence, $\mathfrak{F}_B$ is the set of real numbers

1. $M = \{(-2, 5), (-2, -2), (0, 1), (0, 5), (5, 0), (5, -2)\}$
2. $N = \{(3, 4), (4, 3)\}$
3. T, whose graph is on page 5-B.

4. $R = \{(7, 2), (7, 9), (7, 6), (7, 7)\}$

5. $S = \{(8, 1), (6, 1), (4, 1), (12, 1), (2, 1)\}$

6. $C = \{(x, y): y^2 = 2x - 3\}$

7. $D = \{(x, y): x^2 + y^2 = 25\}$

8. $E = \{(x, y) \in I \times I: |x| + |y| \leq 10\}$

9. $F = \{(x, y): y = |x - 2| + 4\}$

10. $G = \{(x, y): y + x = x + y + 3\}$

11. $H = \{(x, y): xy = yx\}$

12. $J = \{(x, y): x^2 - y^2 = 25\}$

13. $K = \{(x, y): 9x^2 + 25y^2 = 225\}$

[Supplementary exercises are in Part E, page 5-242.]

B. True or false?

Sample. $\exists_x\, x^2 - 5x + 6 = 0.$

Solution. True. [For instance, $3^2 - 5 \cdot 3 + 6 = 0$.]

1. $\exists_y\, 3y + 7 = 18$

2. $\exists_k\, k = k + 1$

3. $\exists_t\, t^2 - 1 = 0$

4. $\exists_m\, 3 + m = m + 3$

5. $\exists_x\, (2x - 5 = 0 \text{ and } 5x - 2 = 0)$

6. $\exists_x\, (2x - 5 = 0 \text{ or } 5x - 2 = 0)$

7. $\forall_x\, 2x - 3 = 12$

8. $\exists_x\, 2x - 3 = 12$

9. $\exists_{x \in I}\, 12 = 3x$

10. $\exists_{x \in I}\, 12 = 7x$

11. $\exists_y\, y^5 - 3y^4 = y(y - 1)(y + 9)$

12. $\forall_x\, (\exists_y\, x + y = 3)$

13. $\exists_y\, (\forall_x\, x + y = 3)$

14. $\forall_x\, (\exists_y\, xy = 0)$

15. $\exists_y\, (\forall_x\, xy = 0)$

C. If P and Q are relations then $P \cup Q$ and $P \cap Q$ are relations. [Why?] Can you compute the domain of $P \cup Q$ if you know $\mathcal{D}_P$ and $\mathcal{D}_Q$? How about the domain of $P \cap Q$? How about ranges and fields?

D. 1. (a) Use the same chart to draw the graphs of the relations A and B where

$$A = \{(1, 4),\ (2, 7),\ (3, 3),\ (4, 2),\ (5, 10)\}$$

and $B = \{(4, 1),\ (3, 3),\ (7, 2),\ (10, 5),\ (2, 4)\}$.

Draw little circles around the dots of the graph of A, and draw little crosses through the dots of the graph of B.

(b) Do you see how the members of B are related to those of A?

2. (a) Draw the graph of C where

$$C = \{(1, 5),\ (3, 7),\ (4, 6),\ (2, 8),\ (5, 9)\}.$$

(b) On the same chart draw the graph of the relation D whose members are related to those of C in the same way that those of B are related to those of A.

3. (a) Draw the graph of E where $E = \{(x, y):\ y - 2x = 1\}$.

(b) Draw the graph of the relation F whose members are the ordered pairs obtained by "reversing" the components of the ordered pairs in E.

4. The relation B of Exercise 1 is called <u>the converse of the relation A</u>. Also, D is the converse of C, and F is the converse of E. For each of the relations listed below, write a name for the relation which is its converse.

(a) $\{(6, 2),\ (9, 9),\ (8, 3)\}$ (b) $\{(x, y):\ y = 3x\}$

(c) $\{(x, y):\ x = y\}$ (d) $\{(x, y):\ y = \frac{1}{3}x\}$

(e) $\{(x, y):\ x^2 + y^2 = 25\}$ (f) $\{(x, y):\ x = 2\}$

(g) the relation of being a child of

(h) the relation of being a cousin of (i) the relation >

5. (a) Suppose the relation S is the converse of the relation R. If $\mathfrak{D}_R = J$ and $\mathfrak{R}_R = K$, what are $\mathfrak{D}_S$, $\mathfrak{R}_S$, and $\mathfrak{F}_S$?

6. Give a definition of the converse of a relation. [<u>Hint.</u> The converse of R is $\{(x, y) \in$ _____: ________________$\}$.]

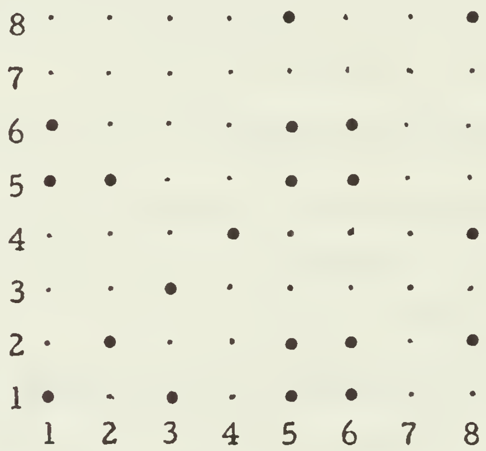
[Supplementary exercises are in Part F, page 5-243.]

REFLEXIVE RELATIONS

In an earlier section we noted that the relation of the inch-measure of one side to the inch-measure of another side of a triangle whose third side is 2 inches long is not a reflexive relation. And, the reason it is not reflexive is that it does not contain ordered pairs like (1, 1) and (0.5, 0.5). These are ordered pairs with equal components which are in the cartesian square of the field of the relation but are not in the relation itself.

A relation R is reflexive if and only if,
for each $x \in \mathfrak{F}_R$, $x \, R \, x$ [that is, $(x, x) \in R$].

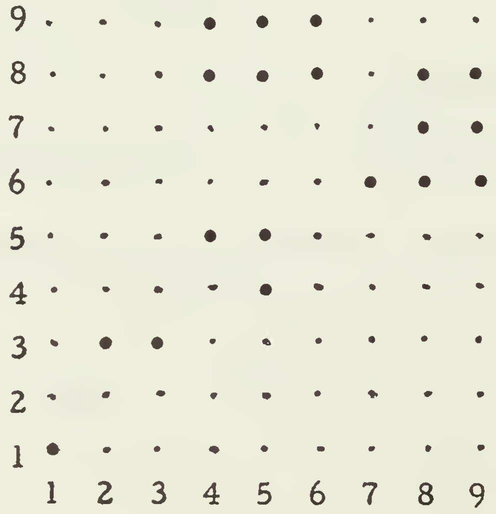
Here is a graph of a relation. Is it reflexive? Justify your answer.



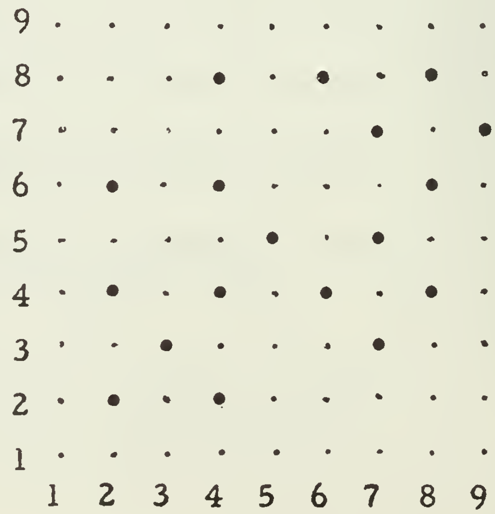
EXERCISES

A. Each exercise contains the graph of a relation. What additional ordered pairs must you include in the relation in order to obtain a reflexive one?

(1)



(2)



B. 1. Suppose R is a reflexive relation and $\mathfrak{D}_R = \{1, 2, 3, 4, 5\}$. What ordered pairs are you sure belong to R?

2. Suppose R is a reflexive relation and $\mathfrak{D}_R = \{3, 4, 5\}$. Can you tell what $\mathfrak{R}_R$ is?

3. If you know that a relation is reflexive, what can you say about its domain and range?

4. Is the converse of a reflexive relation reflexive?

C. Which of these relations are reflexive?

1. $\{(3, 7), (8, 2), (8, 8), (3, 3), (2, 8), (2, 2), (7, 7)\}$
2. $\{(4, 1), (1, 1), (6, 4), (6, 6)\}$
3. $\{(x, y): x = y\}$
4. $\{(x, y): y \leq x\}$
5. $\{(x, y): |x| \leq 5 \text{ and } |y| \leq 5\}$
6. $\{(x, y): x^2 + y^2 \leq 25\}$
7. $\{(x, y) \in I^+ \times I^+: y \text{ is a factor of } x \text{ with respect to } I^+\}$
8. $\{(x, y) \in I \times I: \exists_{q \in I}\ y - x = 5q\}$
9. $\{(x, y) \in P \times P: y \text{ has the same parents as } x\}$, where P is the set of all people.
10. $\{(x, y) \in T \times T: y \text{ has the same perimeter as } x\}$, where T is the set of all triangles.
11. $\{(x, y) \in T \times T: y \text{ has the same shape as } x\}$
12. $\{(x, y) \in P \times P: y \text{ is a sister of } x\}$
13. $\{(x, y) \in Q \times Q: y \text{ has the same roots as } x\}$, where Q is the set of all equations in one variable [equations like '3p - 5 = 2 - 7p' but not like '3a + 6b - 7 = 8a'].
14. $\{(x, y) \in P \times P: y \text{ has the same uncles as } x\}$
15. $\{(x, y): (x - y)^2 + (y - x)^4 = 0\}$

* * *

Given a set S which contains 3 elements, how many subsets does S have? One way of answering this question is just to list the subsets of S and count them. For example:

if $$S = \{a,\ b,\ c\}$$

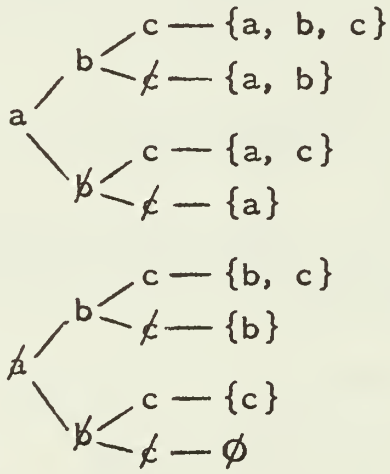
then the subsets of S are

$$\left.\begin{array}{l} \{a,\ b,\ c\} \\ \{b,\ c\},\ \{c,\ a\},\ \{a,\ b\} \\ \{a\},\ \{b\},\ \{c\}, \\ \emptyset. \end{array}\right\} \begin{array}{l} 1 \\ 3 \\ 3 \\ \underline{1} \\ 8 \longleftarrow \text{Total} \end{array}$$

and

But, this would be a tedious method if you wanted to find the number of subsets of a set containing, say, 25 elements.

Let's use another method which is easily generalized. Choosing a subset of S amounts to making a sequence of choices, one for each element of S. One decides for each element if it is to be included in the subset or not. There are two possible outcomes of the first choice. Then, for each of these, there are two possible outcomes of the second choice, etc. For our set S, here is a diagram of the procedure:



Each of the first three columns corresponds to a choice; for example, 'a' indicates that a is chosen, '$\not a$' that a is rejected. The fourth column lists the subsets obtained by various sequences of choices. Notice that the first column has 2 entries, the second column has 2×2 entries, and the third column has $(2 \times 2) \times 2$ entries. Each column of entries after the first has twice as many entries as the preceding column. So, the third column has 2^3 entries; hence, a set with 3 elements has 2^3 subsets.

* * *

D. 1. How many subsets has a set of four elements? A set of five elements? A set of 25 elements?

2. How many relations are there among the members of a set of four elements? [Hint. How many ordered pairs belong to the cartesian square of a set of four elements?]

3. How many relations among the members of {1, 2, 3, 4} contain both the ordered pairs (2, 4) and (4, 1)?

4. If R is a reflexive relation whose field consists of four elements, what can you say about the number of elements in R?

5. How many reflexive relations are there whose field is a given set of four elements?

☆6. How many reflexive relations are there among the members of a given set of four elements? [Hint. Besides those you have counted in Exercise 4, you must take account of reflexive relations whose fields are 3-membered subsets of the given set, 2-membered subsets, etc.]

☆7. How many relations among the members of a given set of four elements have this set as their domain?

E. If you solved Exercise 6 of Part D, you found that a 4-membered set has

	1	0-membered subset [$\emptyset$],
	4	1-membered subsets,
	6	2-membered subsets,
	4	3-membered subsets,
and	1	4-membered subset [the set itself].

Perhaps you did this by considering a particular 4-membered set, say {a, b, c, d}, listing its subsets, and counting them. Just as there is an easy method for finding the total number of subsets of a given set, there are also easy methods for finding the number of those subsets [of a given set] which have a given number of elements.

For example, there are easy methods for finding the number of 6-membered subsets of a 10-membered set, the number of 13-membered subsets of an 18-membered set, etc. You will learn one such method in the exercises which follow.

1. Complete the rows labeled '2', '3', '5', and '6' in the table below. [The row labeled '4' lists, for each whole number k, the number of k-membered subsets of a 4-membered set.]

n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2									
3									
4	1	4	6	4	1	0	0	0	0
5									
6				20					
7									
8									

2. Study the table carefully. Do you see a quick way of getting the numbers listed in, say, the 5-row, from the numbers listed in the 4-row? Can you get those in the 4-row from those in the 3-row? Those in the 6-row from those in the 5-row? Use this quick way, if you find it, to fill the 7-row and the 8-row. [Clearly, the table can be extended indefinitely. If you disregard the '0'-entries in such a table, the remaining entries form a triangular array called Pascal's Triangle. Almost every encyclopedia gives some account of the fascinating life of the seventeenth century French mathematician Blaise Pascal.]

SYMMETRIC RELATIONS

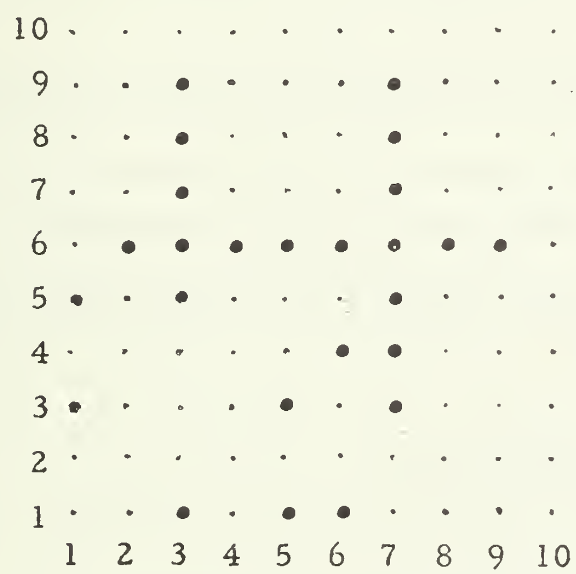
If John is a cousin of Ruth then Ruth is a cousin of John. More generally, we say that the relation of cousinhood is a symmetric relation. Is brotherhood a symmetric relation? [Explain your answer.]

A relation R among the members of a set S is symmetric if and only if, for each $(x, y) \in S \times S$, if $y\ R\ x$ then $x\ R\ y$.

EXERCISES

A. Each exercise contains the graph of a relation. What additional ordered pairs must you include in the relation in order to obtain a symmetric one?

(1)



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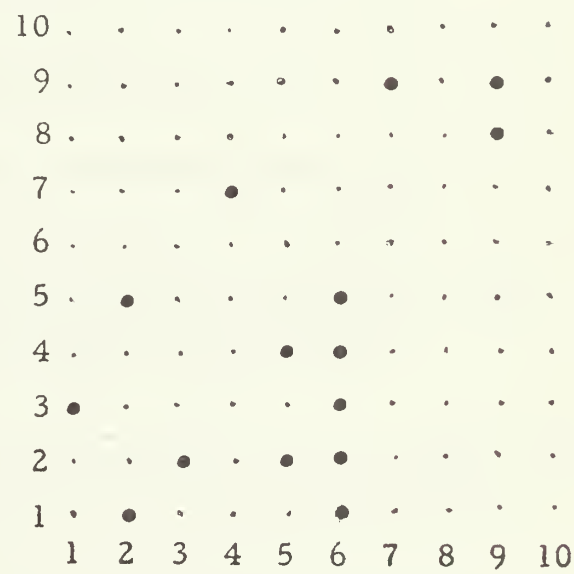

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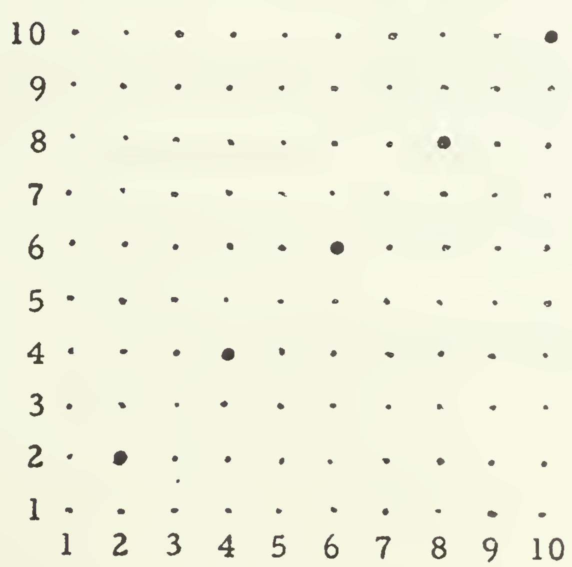
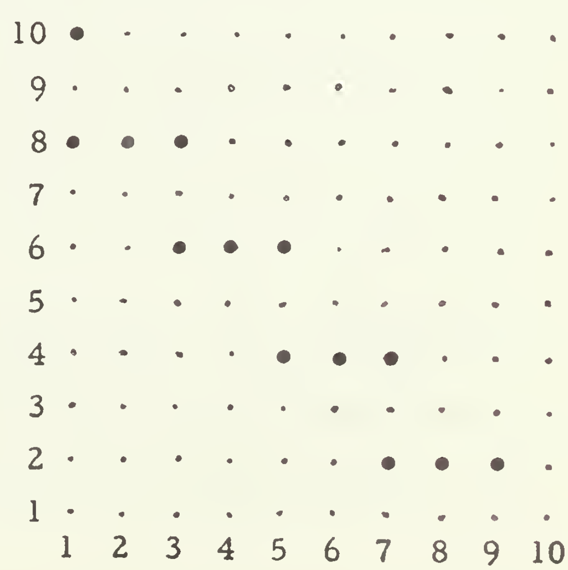


(4)



B. 1. Can a symmetric relation consist of 20 ordered pairs? 21? 1? 0?

2. Can a reflexive relation also be symmetric?

3. Can a relation which is both reflexive and symmetric consist of exactly 3 ordered pairs?

4. Suppose R is a symmetric relation and $\mathcal{D}_R = \{3, 4, 5\}$. Can you tell what $\mathcal{R}_R$ is?

5. If you know that a relation is symmetric, what can you say about its domain and range? What about its converse?

6. What do you know about a relation which is its own converse?

7. Suppose R is a symmetric relation and $\mathcal{F}_R = \{1, 2, 3, 4, 5\}$. What ordered pairs are you sure belong to R? What can you say about the number of members of R?

8. How many symmetric relations are there among five elements?

9. How many relations are there which are both symmetric and reflexive, and whose field is a given set of five elements?

☆10. How many relations are there among five elements which are both reflexive and symmetric?

C. Which of these relations are symmetric?

1. $\{(3, 8), (7, 6), (6, 6), (6, 7), (5, 5), (8, 5), (8, 3), (5, 8)\}$

2. $\{(4, 4), (1, 1), (0, 0), (-2, -2), (6, 4), (6, 6), (4, 4)\}$

3. $\{(x, y): y = 2x + 5\}$

4. $\{(x, y): y = 2x + 5 \text{ or } x = 2y + 5\}$

5. $\{(x, y): y + x = 5\}$

6. $\{(x, y): y = 3x - 2 \text{ or } y = \frac{x + 2}{3}\}$

7. $\{(x, y): xy < 0\}$

8. $\{(x, y): x^2 + y^2 = 25\}$

9. $\{(x, y): 4x^2 + 5y^2 = 101\}$

10. $\{(x, y): y = x - 1\}$

11. $\{(x, y): x^2 + xy + y^2 = 10\}$

12. $\{(x, y): y + x + 1 = 0 \text{ or } y - x + 1 = 0\}$

13. $\{(x, y): y - 1 < x < y + 1\}$

D. 1. Which of the relations listed in Part C on page 5-46 are reflexive?

2. Which of the relations listed in Part C on page 5-41 are symmetric?

E. Each of the exercises below describes a relation among geometric figures in a plane. For each exercise,

(a) make a picture of the components of an ordered pair which belongs to the relation,

(b) tell whether the relation is reflexive,

(c) tell whether it is symmetric, and

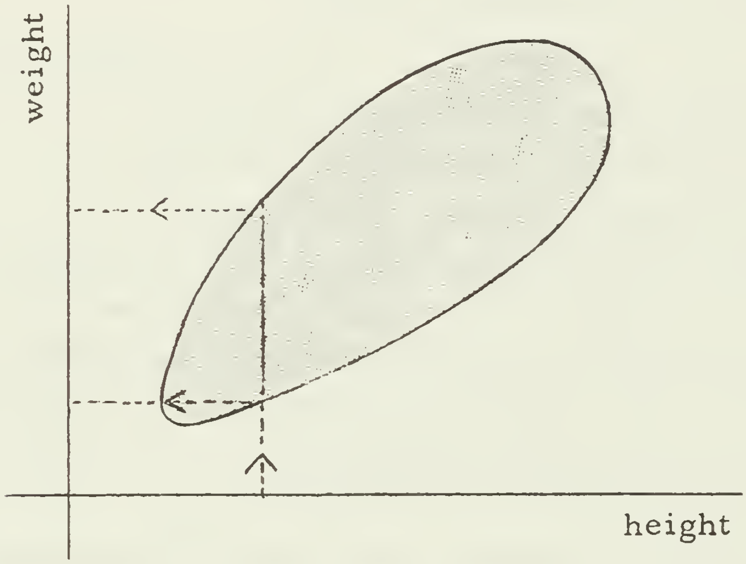
(d) give a commonly used name for the relation.

[T = the set of all triangles; C = the set of all circles; S = the set of all segments; L = the set of all [straight] lines; A = the set of all geometric figures]

1. $\{(x, y) \in T \times T$: x has the same size and shape as y$\}$
2. $\{(x, y) \in L \times L$: $x \cap y = \emptyset\}$
3. $\{(x, y) \in C \times C$: x has the same circumference as y$\}$
4. $\{(x, y) \in S \times S$: x has the same length as y$\}$
5. $\{(x, y) \in L \times L$: $x \cup y$ contains a right angle$\}$
6. $\{(x, y) \in T \times T$: x has the same shape as y$\}$
7. $\{(x, y) \in T \times T$: the set of measures of the sides of x is the set of measures of the sides of y$\}$
8. $\{(x, y) \in T \times T$: the set of measures of the angles of x is the set of measures of the angles of y$\}$
9. $\{(x, y) \in A \times A$: x has the same size and shape as y$\}$
10. $\{(x, y) \in A \times A$: x has the same shape as y$\}$
11. $\{(x, y) \in T \times T$: x has the same perimeter as y$\}$
12. $\{(x, y) \in T \times T$: x has the same area as y$\}$

[Supplementary exercises on symmetry and reflexivity of relations are in Part G, pages 5-244 through 5-245, and in Part I, page 5-248. Optional exercises on other properties of relations are in Part H, pages 5-245 through 5-247, and in Part J, pages 5-249 through 5-250.]

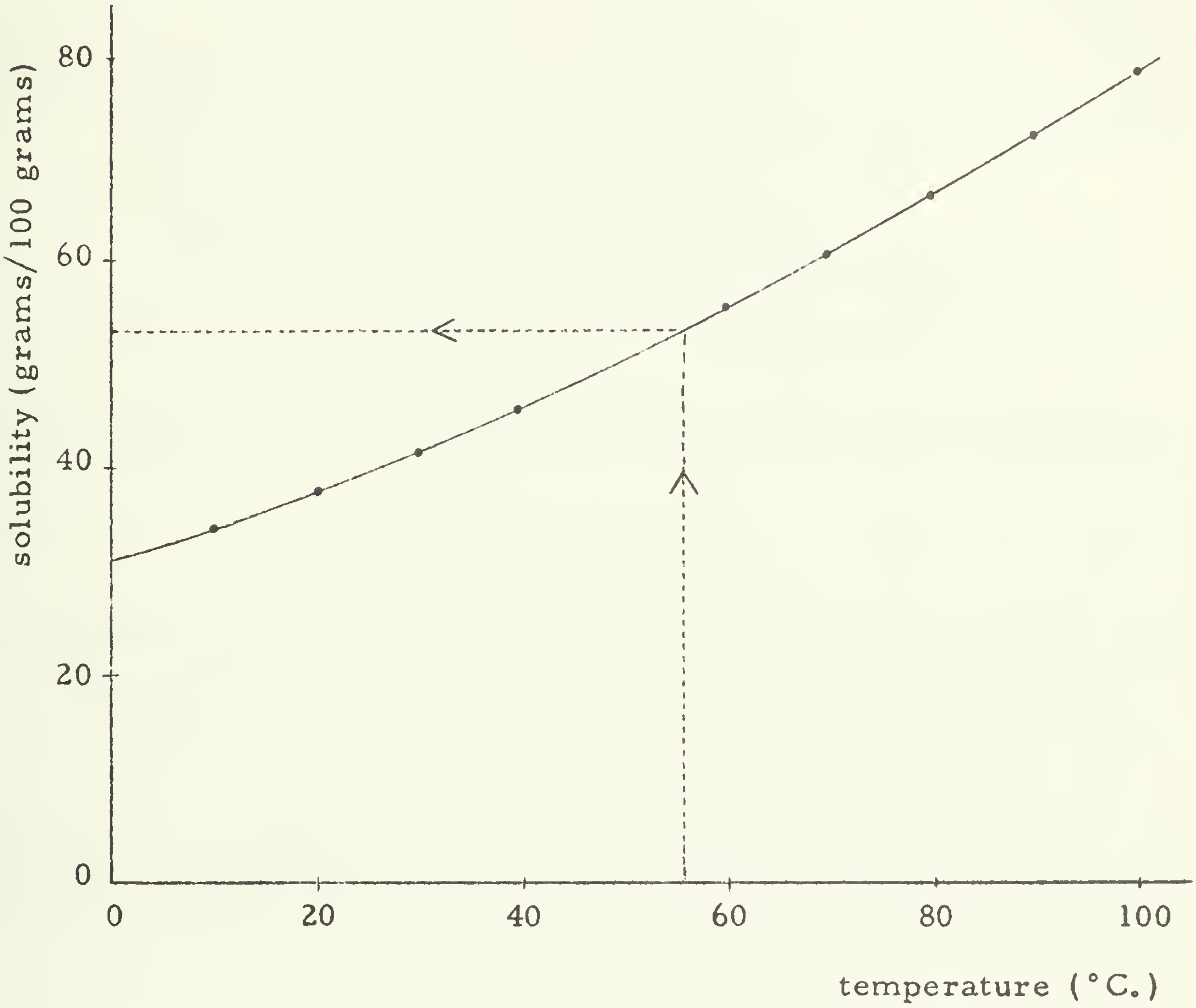
5.05 Functions.--We started our discussion of relations by pointing out that scientific investigators often ask questions about the relationship of one thing to another. For example, suppose an anthropologist wants to know the relation of weight to height for a given group of people. After collecting data [ordered pairs], he might get this picture of the relation:



If he has reason to believe that the group he worked with is representative of a larger group, he can use this chart to make certain predictions about individuals in the larger group. For example, given the height of a person, he can use the chart to predict upper and lower bounds for the person's weight.

Another investigator, this time a chemist, wants to know the relation of the solubility of a certain salt to the temperature of the water in which the salt is being dissolved. One way of doing this is to take a known quantity of water, say 100 grams, heat it to a certain temperature and keep it there while slowly adding salt crystals to the water until no more salt can be dissolved. If he uses 100 grams of water then the number of grams of salt which can be dissolved is the solubility of the salt at that temperature. To minimize the effect of experimental error, the chemist would repeat the experiment several times for each temperature and average the results. The ordered pairs for ten temperatures are shown by the heavy dots on the chart on the next page. The chemist makes the reasonable guess that if he draws a smooth curve which follows the pattern of these dots, he will have a picture of the

relation of solubility of this salt [ammonium chloride] to temperature.



With this chart, the chemist can predict the solubility of ammonium chloride at any temperature [between 0°C and 100°C]. What solubility would he predict for ammonium chloride at 48°C?

Although both the anthropologist and the chemist discover and graph relations from which predictions can be made, the predictions made by the chemist are of a different kind than those made by the anthropologist. For each temperature, the chemist can predict the corresponding solubility. For each height, the anthropologist can only assign bounds to the set of corresponding weights. The chemist can claim that solubility of ammonium chloride is <u>determined</u> by temperature; the anthropologist cannot claim that weight is determined by height--presumably, other factors are involved.

Relations such as the chemists' in which the value of one quantity determines the corresponding value of another are called functional relations--for short, functions. In graphical terms, a function is a relation such that each vertical line crosses its graph in at most one point. [Vertical lines which do not cross the graph correspond with elements not in the domain of the relation.] In other words:

> A function is a set of ordered pairs no two of which have the same first component.

All functions are relations, but not all relations are functions.

EXERCISES

A. Here are several sets of ordered pairs. All of these sets are relations and some are functions. Tell which are functions.

Sample. $\{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (7, 5)\}$ (7, 9)

Solution. This relation is not a function because two of its ordered pairs--(7, 5) and (7, 9)--have the same first component.

1. $\{(3, 7), (8, 4), (6, 5), (10, -7), (-4, 2), (6, -4)\}$

2. $\{(4, 3), (6, 3), (-5, 3), (-2, 3), (5, 3), (2, 3)\}$

3. $\{(-1, 6), (-1, 4), (-1, 8), (-1, 8), (-1, 64), (-1, 3)\}$

4. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (1.5, 2.5), (2, 4), (7, \pi)\}$

5. $\{(3, 2), (2, 3), (-2, 3), (-3, 2), (4, 3), (3, \sqrt{4})\}$

6. $\{(5, 7), (-3, 7), (59, 7), (\pi, 7), (803, 7)\}$

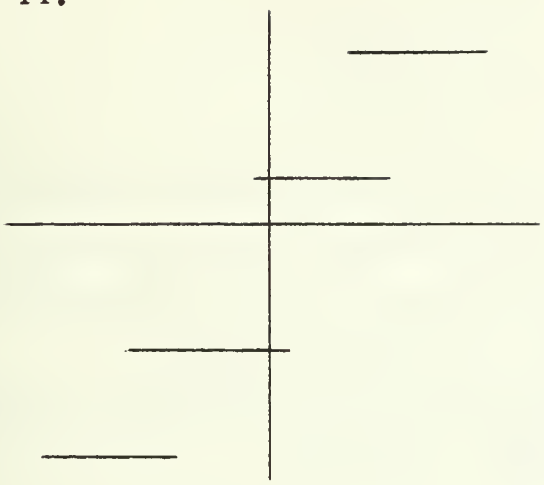
7. $\{(3, 8)\}$

8. $\emptyset$

9. $\{(x, y): y = 2x + 1\}$

10. $\{(x, y): x^2 + y^2 = 5\}$

11.



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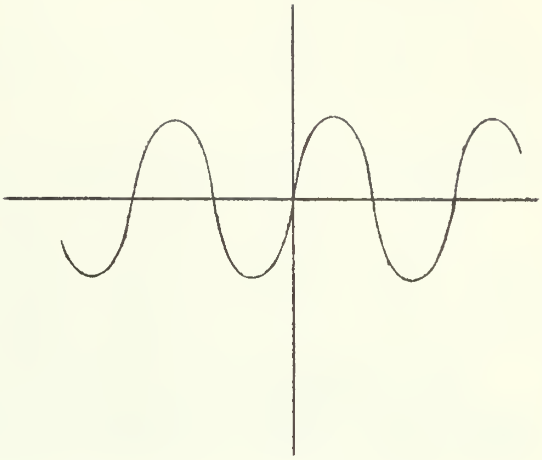

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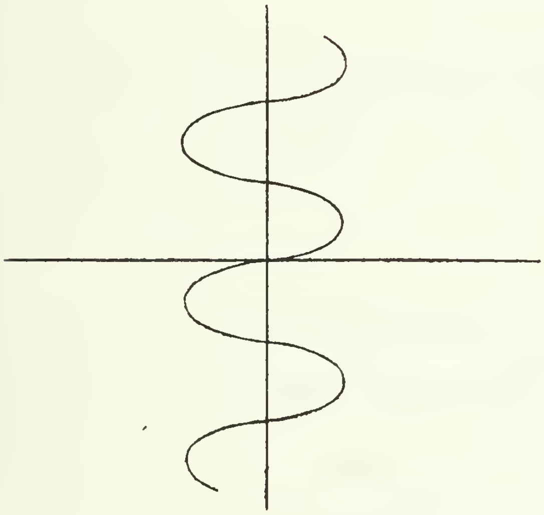

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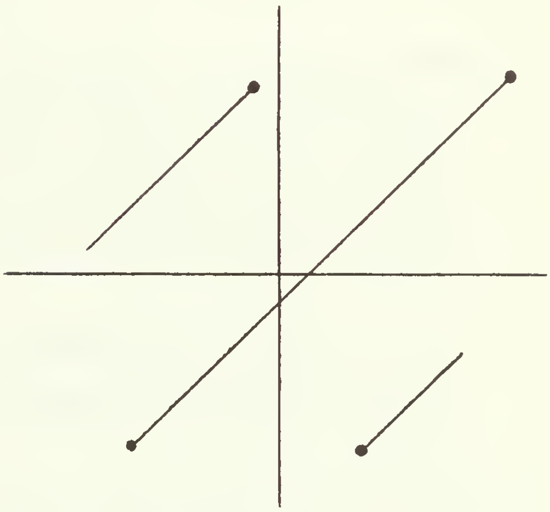

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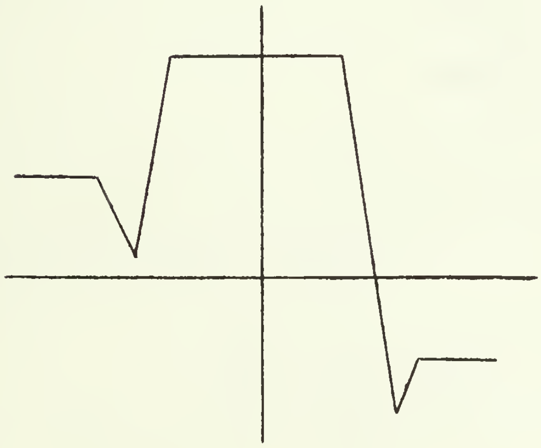

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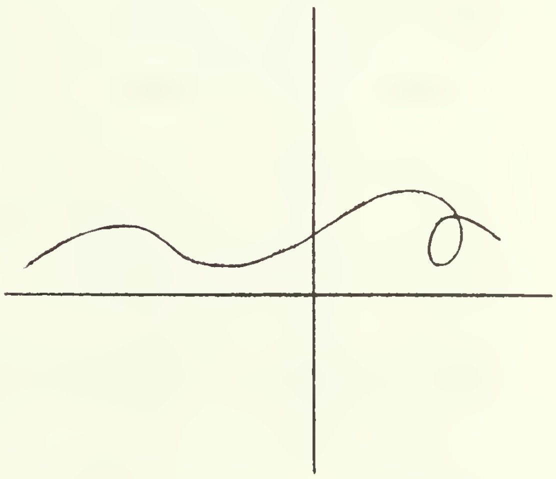


B. Which of the relations in Part A have converses which are functions?

* * *

Consider the functions f and g where

$$f = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$$

and $$g = \{(2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}.$$

Is it the case that $f(2) = g(2)$? That $f(3) = g(3)$? That $f(4) = g(4)$?

Is it the case that, for each x which belongs to both $\mathcal{D}_f$ and $\mathcal{D}_g$, $f(x) = g(x)$? [If you answer 'no' to this question, it means that there is an $x \in \mathcal{D}_f \cap \mathcal{D}_g$ such that $f(x) \neq g(x)$. Is this the case?]

Even though f and g have the same value for each of their common arguments, they are different functions. Explain.

* * *

C. For each pair of functions, answer these three questions:

(a) Is it the case that, for each $x \in \mathcal{D}_f \cap \mathcal{D}_g$, $f(x) = g(x)$?

(b) Is it the case that $\mathcal{D}_f = \mathcal{D}_g$?

(c) Is it the case that $f = g$?

1. $f = \{(\text{Tim}, 3), (\text{Bill}, 4), (\text{Ed}, 2), (\text{John}, 4)\}$
   $g = \{(\text{Tim}, 3), (\text{Mary}, 4), (\text{John}, 4), (\text{Cal}, 3)\}$

2. $f = \{(x, y), x = 0, 1, 2, 3, 4: y = x + 1\}$
   $g = \{(x, y), x = 2, 3, 4, 5, 6: y = x + 1\}$

3. $f = \{(4, 7), (2, 9), (3, 5), (6, 7)\}$
   $g = \{(1, 8), (5, 7), (8, 3), (9, 2)\}$

4. $f = \{(x, y): y = x + 4\}$ $\quad g = \{(x, y), x \neq 0: y = x + 4\}$

5. $f = \{(x, y): xy = 1\}$ $\quad g = \{(x, y), x \neq 0: y = \frac{1}{x}\}$

6. $f = \{(x, y), 2 \leq x \leq 6: y = x^3 - 5x^2 + 7x - 2\}$
   $g = \{(x, y), 1 \leq x \leq 5: y = x^3 - 5x^2 + 7x - 2\}$

7. $f = \{(x, y): y = \sqrt{x^2}$ $\quad g = \{(x, y): y = |x|\}$

*

8. If f and g are functions such that $\mathcal{D}_f = \mathcal{D}_g$ and $\mathcal{R}_f = \mathcal{R}_g$, does it follow that $f = g$? Explain.

D. Although many of the functions you will work with in mathematics are sets of ordered pairs of real numbers, functions can be sets of ordered pairs of any kind. For example, the domain of a function can consist of wagons, and its range of horses. Each of the following exercises refers to a relation. Tell if the relation is a function and be prepared to support your answer.

Sample 1. $\{(x, y) \in \text{People} \times \text{People}:\ x \text{ is the father of } y\}$

Solution. Since some fathers have more than one child, this relation is not a function. [Note that 'People' is being used as a name for the set of all human beings.]

Sample 2. $\{(x, y) \in \text{Circles} \times \text{Points}:\ y \text{ is the center of } x\}$

Solution. Since each circle has at most one center, this is a function. [Since each circle has a center, the domain of this function is the set of all circles. Do you think that $\{(x, y) \in \text{Points} \times \text{Circles}:\ x \text{ is the center of } y\}$ is a function?]

1. $\{(x, y) \in \text{People} \times \text{People}:\ x \text{ is a son of } y\}$

$\{(x, y) \in \quad \times$: y is a son

2. $\{(x, y) \in \text{People} \times \text{People}:\ x \text{ is a son of } y \text{ and } y \text{ is a woman}\}$

3. $\{(x, y) \in \text{New Yorkers} \times \text{Whole numbers}:\ x \text{ owns } y \text{ automobiles}\}$
[Discuss the range of this relation.]

4. $\{(x, y) \in \text{Students} \times \text{Students}:\ x \text{ and } y \text{ are in the same grade}\}$

5. $\{(x, y) \in \text{Days} \times \text{Whole numbers}:\ \text{there were } y \text{ traffic deaths on } x\}$

6. $\{(x, y) \in \text{Real numbers} \times \text{Integers}:\ y \le x < y + 1\}$
[Graph this relation.]

7. $\{(x, y) \in C_1 \times \text{Reals}:\ y \text{ is the first component of the point } x\}$
[C_1 is the unit circle, that is, the circle in the number plane with center (0, 0) and radius 1.]

8. $\{(x, y) \in \text{Reals} \times C_1:\ \text{the point } y \text{ has } x \text{ for first component}\}$

9. $\{(x, y) \in \text{Houses} \times \text{Whole numbers}:\ \text{there are } y \text{ rooms in } x\}$

10. $\{(x, y) \in \text{Squares} \times N:\ y \text{ is the area-measure of } x\}$
[N is the set of numbers of arithmetic.]

11. $\{(x, y) \in \text{Squares} \times N$: y is the side-measure of $x\}$

12. $\{(x, y) \in \text{Males} \times N$: x is 20 years old and is y inches tall$\}$

13. $\{(x, y) \in \text{Triangles} \times \text{Circles}$: y contains the vertices of $x\}$

14. $\{(x, y) \in \text{Circles} \times \text{Triangles}$: x contains the vertices of $y\}$

15. $\{(x, y) \in \text{Ears of corn} \times \text{Whole numbers}$: x has y rows of kernels$\}$

16. [Suppose a biologist is conducting a week-long nutrition experiment on the members of a set of hamsters. At noon on each day he records the weight and the blood pressure of each hamster. Let H be the set of hamsters and D be the set of days of the experiment.]

    (a) $\{(x, y) \in D \times N$: Hamster 7 weighs y ounces on $x\}$

    (b) $\{(x, y) \in D \times N$: y is the average of the ounce-measures of the weights of the hamsters on $x\}$

    (c) $\{(x, y) \in H \times N$: x weighs y ounces on Thursday$\}$

    (d) $\{(x, y) \in H \times N$: y is the measure of the blood pressure of x on Wednesday$\}$

    (e) $\{(x, y) \in D \times N$: y is the measure of the blood pressure of Hamster 3 on $x\}$

    (f) $\{(x, y) \in N \times N$: $\exists_{z \in D}$ x is the measure of the weight of Hamster 7 on z and y is the measure of its blood pressure on $z\}$

17. [Assume that no person has more than a million hairs on his head.]

    (a) $\{(x, y) \in \text{New Yorkers} \times \text{Whole numbers}$: x has y hairs on his head$\}$

    (b) $\{(x, y) \in \text{Whole numbers} \times \text{New Yorkers}$: y has x hairs on his head$\}$

    (c) Show that in a class of 25 students at least 3 have birthdays falling in the same month.

☆ 18. $\{(x, y) \in \text{Rectangles} \times (N \times N)$: the components of y are the inch-width and the inch-length of x, respectively$\}$

E. The relations in Part D which are functions are commonly referred to by noun phrases such as 'the center of a circle' [Sample 2] and 'a male person's mother' [Exercise 2]. We give below several noun phrases which refer to some of the functions given in the exercises of Part D. Name the exercise in the blank following the noun phrase.

1. the number of rooms in a house ________________
2. the number of automobiles owned by a New Yorker __________
3. the greatest integer not greater than a real number __________
4. the circle which circumscribes a triangle ________________
5. the first component of a point on the unit circle _____________
6. the area-measure of a square ________________
7. the daily traffic death toll ________________
8. the number of hairs on a New Yorker's head ______________
9. the daily measure of blood pressure of Hamster 3 ___________
10. the number of rows of kernels on an ear of corn _________
11. the inch-height of a 20-year old man ____________
12. the ounce-weight of an experimental hamster on Thursday ____
13. the inch-dimensions of a rectangle ___________
14. the side-measure of a square _____________

F. Write brace-notation names for the functions referred to by the noun phrases.

Sample 1. the radius of a circle

Solution. $\{(x, y) \in \text{Circles} \times N\colon\ y \text{ is the radius of } x\}$

Sample 2. the sum of a real number and 2

Solution. $\{(x, y)\colon\ y = x + 2\}$

1. the number of children in a family
2. the perimeter of a triangle
3. the diameter of a circle
4. the dollar income of a man during his lifetime
5. the annual dollar income of John E. Wilson
6. the number of students in a Zabranchburg High class

7. the number of inches of annual rainfall in Portland, Oregon
8. the daily temperature of a certain frog
9. the number of trials it takes a rat to learn a certain maze
10. the number of trials it takes a certain rat to learn a maze
11. the square root of a nonnegative number
12. the opposite of a real number
13. the square of a number
14. the number of prime factors of a positive integer

FUNCTIONAL NOTATION

Since a function is a relation, a function has a domain, and a range. The members of the domain of a function are often called <u>arguments of the function</u>, and the members of its range are called <u>values of the function</u>. Functions can be named just as one names other relations.

By definition, for each argument x of a function F, there is one and only one value y of F such that $(x, y) \in F$. Suppose F is $\{(x, y): y = 2x - 1\}$. Now, 3 is an argument of F, and the y such that $(3, y) \in F$ is $2 \cdot 3 - 1$, or 5. Instead of saying that

$$(3, 5) \in F \quad \text{or that} \quad 5\ F\ 3,$$

it is customary to say that

$$F(3) = 5.$$

[Read this as 'F of 3 is 5'.] So, in this context, the symbol 'F(3)' is a numeral for 5. [Of course, when 'F' is used as a name of a different function, 'F(3)' may be a name for something else, and 'F(3)' will be nonsense if 3 is not an argument of the function.]

Thus, one can form a name for a value of a function by combining a name of the function with a name of a corresponding argument. Such a procedure does not work with relations which are not functions. For example, suppose $R = \{(x, y): x^2 + y^2 = 25\}$. In general, it is not possible to specify a member of the range of R merely by specifying a member of its domain. It would be nonsense to say, for example, that 'R(3)' is a name for <u>the</u> member of the range of R which corresponds with 3. Why?

EXERCISES

A. Each of these functions has a domain which contains the argument 3. Tell the value of the function which corresponds with this argument.

1. $\{(5, 8), (3, 4), (16, 8), (17, 4)\}$
2. $\{(7, 3), (3, 8), (8, 3), (51, 3)\}$
3. $\{(x, y): x + y = 7\}$ 4. $\{(x, y): x^3 + y = 5\}$

B. In each of the following exercises you are given a function. Some of the values of the function are given in functional notation. Your job is to use ordinary notation to tell what these values are.

Sample 1. $g = \{(5, 9), (7, 3), (3, 7), (8, 4)\}$

(a) $g(5) =$ ______ (b) $g(7) =$ ______
(c) $g(3) =$ ______ (d) $g(13) =$ ______
(3) $\mathcal{D}_g =$ ____________ (f) $\mathcal{R}_g =$ ____________

Solution. (a) $g(5) = 9$ (b) $g(7) = 3$
(c) $g(3) = 7$
(d) '$g(13)$' is nonsense because 13 is not in the domain of g.
(e) $\mathcal{D}_g = \{3, 5, 7, 8\}$ (f) $\mathcal{R}_g = \{3, 4, 7, 9\}$

1. $f = \{(1, 1), (2, 4), (3, 11), (4, 18), (5, 27)\}$

(a) $f(4) =$ ________ (b) $f(1) =$ ________
(c) $f(5) =$ ________ (d) $f(11) =$ ________
(e) $\mathcal{D}_f =$ ____________________ (f) $\mathcal{R}_f =$ ____________________

2. $F = \{(1.1, 10), (1.3, 9), (1.6, 8), (2, 7), (1.3, \sqrt{81})\}$

(a) $F(1.6) =$ ________ (b) $F(1.2) =$ ________
(c) $F(2) =$ ________ (d) $F(1.3) =$ ________
(e) $\mathcal{D}_F =$ __________________ (f) $\mathcal{R}_F =$ ______________

3. $G = \{(x, y): y = 3x + 1\}$

(a) $G(5) =$ ________ (b) $G(-2) =$ ________
(c) $G(3.3) =$ ________ (d) $G(\pi) =$ ________
(e) $\mathcal{D}_G =$ __________________ (f) $\mathcal{R}_G =$ __________________

4. $f = \{(x, y): \text{ x is an integer and } y = 2x - 7\}$

(a) $f(5) =$ ______ (b) $f(12) =$ ______

(c) $f(3.2) =$ ______ (d) f(___) = 15

(e) f(___) = 14 (f) f(___) = 3.5

(g) $\mathcal{D}_f =$ ______ (h) $\mathcal{R}_f =$ ______

5. $g = \{(x, y): xy = 1\}$

(a) $g(6) =$ ______ (b) $g(1/6) =$ ______

(c) $g(-2) =$ ______ (d) g(___) = 1/2

(e) $g(0) =$ ______ (f) g(___) = 0

(g) $\mathcal{D}_g =$ ______ (h) $\mathcal{R}_g =$ ______

6. $H = \{(x, y): y = 7\}$

(a) $H(2) =$ ______ (b) $H(3) =$ ______

(c) $H(7) =$ ______ (d) $H(506.5) =$ ______

(e) H(___) = 7 (f) H(___) = 0

(g) $\mathcal{D}_H =$ ______ (h) $\mathcal{R}_H =$ ______

7. $G = \{(x, y): \text{ y is an integer and } y \leq x < y + 1\}$

(a) $G(4.3) =$ ______ (b) $G(5) =$ ______

(c) $G(-4.3) =$ ______ (d) $G(-5) =$ ______

(e) $G(0.5) =$ ______ (f) $G(-0.5) =$ ______

(g) G(___) = 3.7 (h) G(___) = 4

(i) $\mathcal{D}_G =$ ______ (j) $\mathcal{R}_G =$ ______

8. $f = \{(x, y): (x \geq 0 \text{ and } y = 1) \text{ or } (x < 0 \text{ and } y = -1)\}$

(a) $f(9) =$ ______ (b) $f(-0.003) =$ ______

(c) $f(0) =$ ______ (d) f(___) = 1

(e) f(___) = 0 (f) f(___) = −1

(g) $\mathcal{D}_f =$ ______ (h) $\mathcal{R}_f =$ ______

9. A = {(Mary, 1), (Ruth, 3), (Martin, 5), (Alice, 8)}

(a) A (Ruth) = ______ (b) A (Mary) = ______

(c) A (____) = 5 (d) A (____) = 3

(d) A (George) = ______ (f) A (Alice) = ______

(g) $\mathcal{D}_A =$ ______ (h) $\mathcal{R}_A =$ ______

Sometimes a function is described by an equation and a statement of the domain of the function. For example, the function f where $f = \{(x, y): y = 3x\}$ is described by: $f(x) = 3x$, $\mathcal{D}_f$ = the set of real numbers.

*

10. $f(x) = 5x$, $\mathcal{D}_f$ = the set of real numbers

(a) $f(5) =$ _____ (b) $f(-2) =$ _____ (c) $f(100) =$ _____

(d) $f(500) =$ _____ (e) $6 + f(3) =$ _____ (f) $1000 \times f(4) =$ _____

(g) $f(\_\_\_\_) = 0$ (h) $f(\_\_\_\_) = -5$ (i) $\mathcal{R}_f =$ __________

11. $F(x) = 4x - 10$, $\mathcal{D}_F = \{x: x > 2.5\}$

(a) $F(1) =$ _____ (b) $F(3) =$ _____ (c) $F(2) =$ _____

(d) $F(-2) =$ _____ (e) $F(10) =$ _____ (f) $F(7) - F(5) =$ _____

(g) $F(3) \div F(2.5) =$ _____ (h) $F(\_\_\_\_) = 71$

(i) $F(\_\_\_\_) = 0$ (j) $\mathcal{R}_F =$ __________

12. $h(x) = x^2 + 3x + 2$, $\mathcal{D}_h$ = the set of real numbers

(a) $h(-5) =$ _____ (b) $h(0) =$ _____ (c) $h(3) + h(-3) =$ ____

(d) $h(3 + -3) =$ _____ (e) $h(3) \div h(4) =$ _____ (f) $h(\_\_\_\_) = 0$

(g) $\mathcal{R}_h =$ ______________ [Hint. Graph h.]

13. $g(x) = \frac{3}{x - 3}$, $\mathcal{D}_g = \{x: x \neq 3\}$

(a) $g(6) =$ _____ (b) $g(0) =$ _____ (c) $g(5) =$ _____

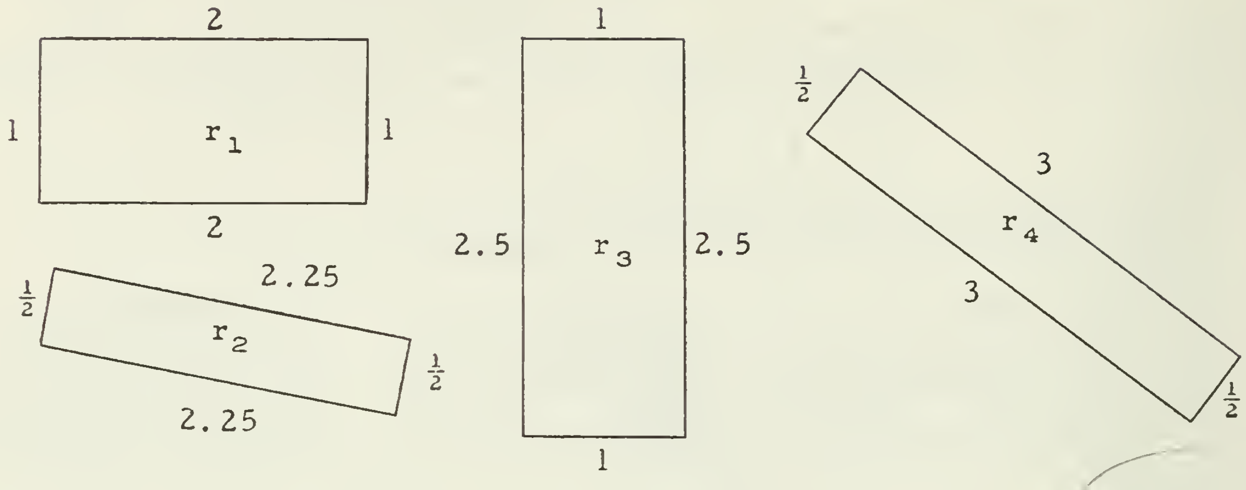
(d) $g(1) =$ _____ (e) $g(1003) + g(-997) =$ _____

(f) $g(487) + g(\_\_\_\_) = 0$ (g) $\mathcal{R}_g =$ ______________

C. In each of the following exercises you are given the domain of a function and a sentence which can be used to compute the values of the function. Your job is to find the range of the function.

1. $\mathcal{D}_f = \{0, 1, 2\}$, $f(x) = 7x^2 - 3x + 2$
2. $\mathcal{D}_f = \{0\}$, $f(x) = -\sqrt{x}$
3. $\mathcal{D}_f$ = the set of integers, $f(x) = x$
4. $\mathcal{D}_f = \emptyset$, $f(x) = 3x + 2$
5. $\mathcal{D}_f$ = the set of real numbers, $f(x) = 9$

D. Here are some rectangles, r_1, r_2, r_3, and r_4.



Consider the functions A and P, where

$$A = \{(x, y) \in R \times N:\ y \text{ is the area-measure of } x\},$$

$$P = \{(x, y) \in R \times N:\ y \text{ is the perimeter of } x\},$$

R is the set of all rectangles, and N is the set of numbers of arithmetic.

1. $P(r_1) =$ ___ 2. $P(r_2) =$ ___ 3. $A(r_1) =$ ___ 4. $A(r_2) =$ ___

5. $P(r_3) =$ ___ 6. $P(r_4) =$ ___ 7. $A(r_3) =$ ___ 8. $A(r_4) =$ ___

9. Consider the relation

$$\{(x, y) \in N \times N:\ \exists_{r \in R}\ x = P(r) \text{ and } y = A(r)\}.$$

Is this relation a function? Is its converse a function? Explain your answers.

10. Consider the functions ℓ and w, where ℓ = the length of a rectangle and w = the width of a rectangle.

(a) Suppose e is a rectangle such that $\ell(e) = 5$ and $w(e) = 2$. Then, $P(e) =$ ____ and $A(e) =$ ____.

(b) If $\ell(e) = 9$ and $A(e) = 36$ then $w(e) =$ ____ and $P(e) =$ ____.

(c) If $\ell(e) = 2w(e)$ and $A(e) = 50$ then $P(e) =$ ____.

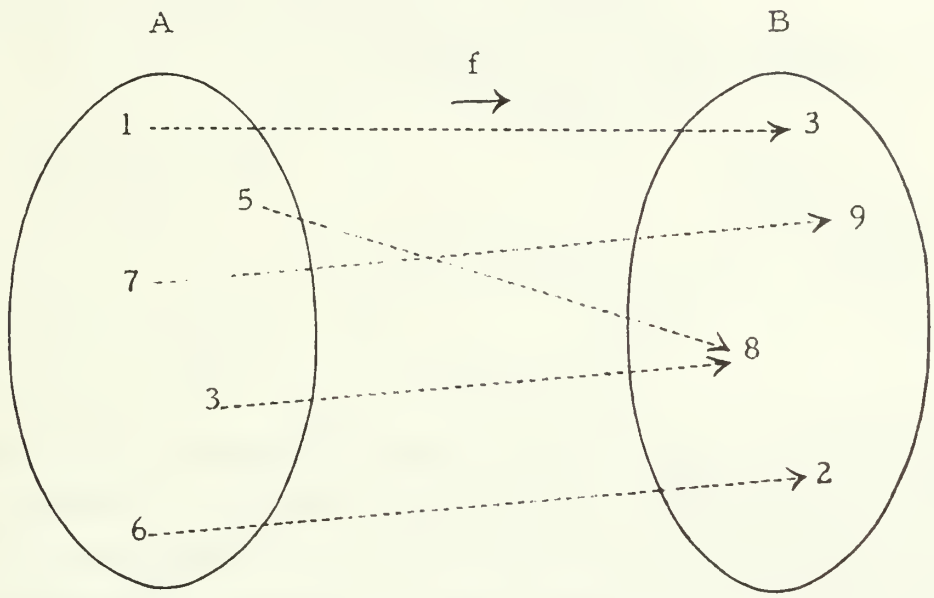
☆(d) If $\ell(e) = w(e) + 5$ and $A(e) = 6$ then $P(e) =$ ____.

☆(e) If $A(e) = 6$ and $P(e) = 10$ then $w(e) =$ ____ and $\ell(e) =$ ____.

[Supplementary exercises are in Part K, pages 5-251 through 5-253.]

FUNCTIONS AS MAPPINGS

Every function has the property that with each member of its domain, A, there corresponds a unique member of its range, B, and each member of B corresponds with some member of A.



$f = \{(1, 3), (5, 8), (7, 9), (3, 8), (6, 2)\}$

Such a correspondence is called a mapping of A on B. It is often convenient to think of a function as determining a mapping of its domain on its range. In the illustration above, we say that f maps $\{1, 3, 5, 6, 7\}$ on $\{2, 3, 8, 9\}$. Also, we call the value of a function for a given argument the image of this argument. So, in the case above, 3 is the image of 1, the image of 7 is 9, 6's image is 2, and 8 is the image of both 3 and 5. [Can an argument of a function have two images?]

If we are given the ordered pairs which belong to a function (either listed between braces, or by means of a table), we can find the image of any member of its domain by hunting up the ordered pair whose first component is the given argument and seeing what its second component is. If a function is described as the set of ordered pairs whose components satisfy a given sentence then, to find the image of an argument, we substitute a name for the argument in the sentence and compute to find [a name for] its image.

If a function is described by a graph, we use the familiar "up-or-down and over-or-back" technique.

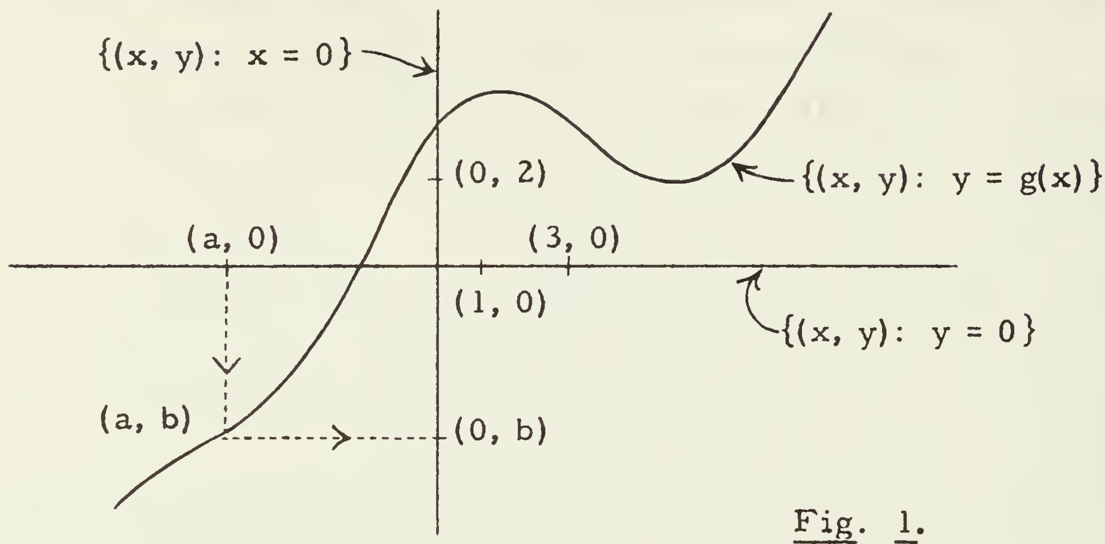


Fig. 1.

As indicated in Figure 1, the function maps the number a of its domain on the number b of its range. In using the graph of g to find the image of a member of the domain, we don't need to view the graphs of g and the axes as pictures of sets of ordered pairs. Instead, we can think of the graph of the x-axis as a picture of the set of real numbers [containing a picture of the domain of g], and we can think of the graph of the y-axis as another picture of the number line [containing a picture of the range of g]. Then, the graph of g is a "mapping line" which we use in finding the image of a member of the domain of g.

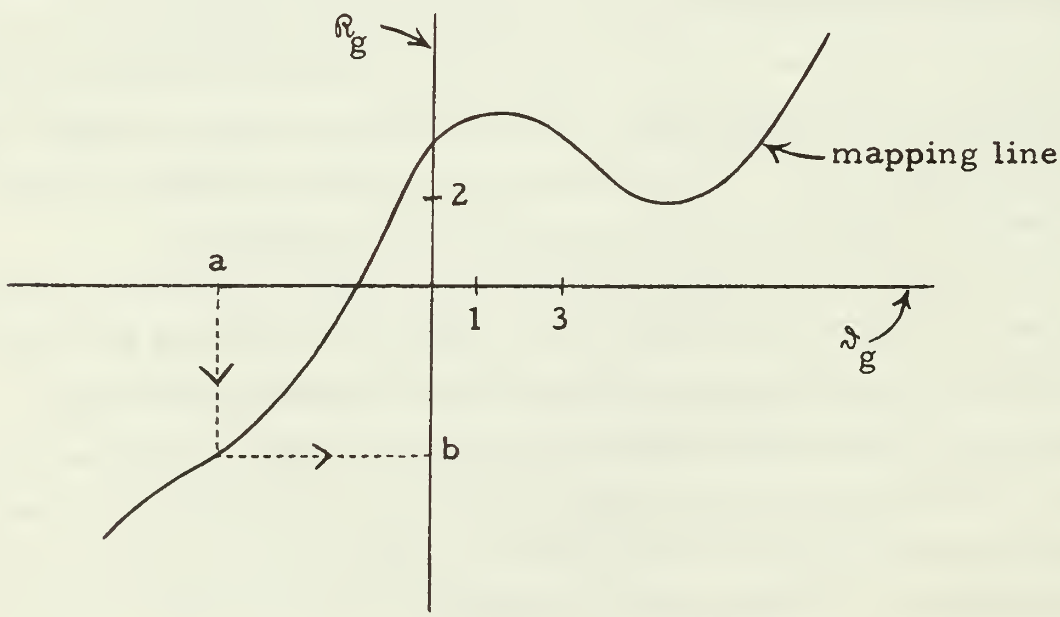


Fig. 2.

Notice that when we used Figure 1, we thought of the curved line as a picture of a set of ordered pairs, and, for that matter, of each point of the paper as the graph of an ordered pair. In Figure 2, only the horizontal and vertical lines are pictures of sets. The only purpose of the rest of the paper is to hold these lines and the mapping line together. Notice also, that it doesn't matter much where we draw pictures of $\mathcal{D}_g$ and $\mathcal{R}_g$. For example, Figure 3 does as well as Figure 2.

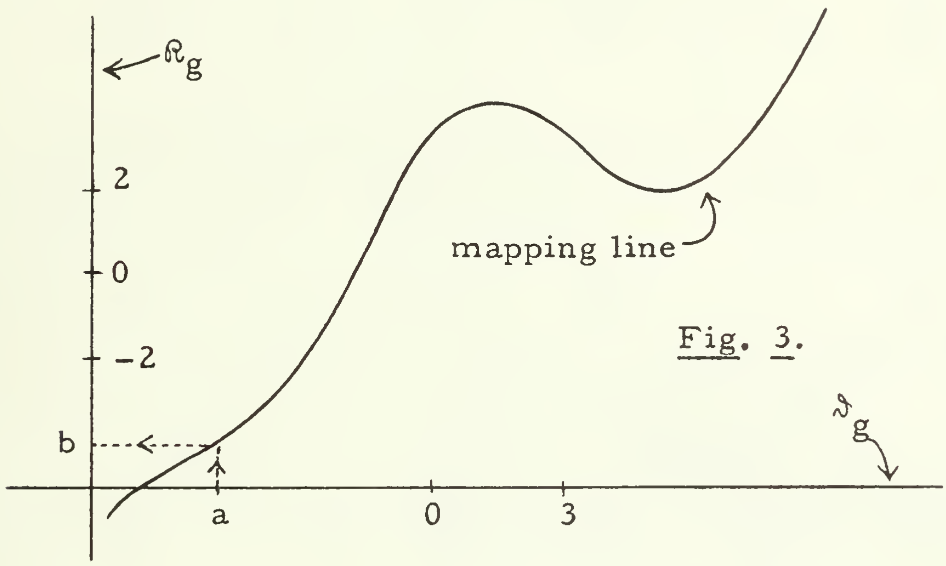


Fig. 3.

Using the graph of a function as a mapping line in determining images of members of the domain of the function is one very handy way of picturing just how the function maps its domain on its range. But there are other ways to do this. For example, Figures 4 and 5 show two ways of picturing the mapping determined by $\{(x, y): y = x + 2\}$.

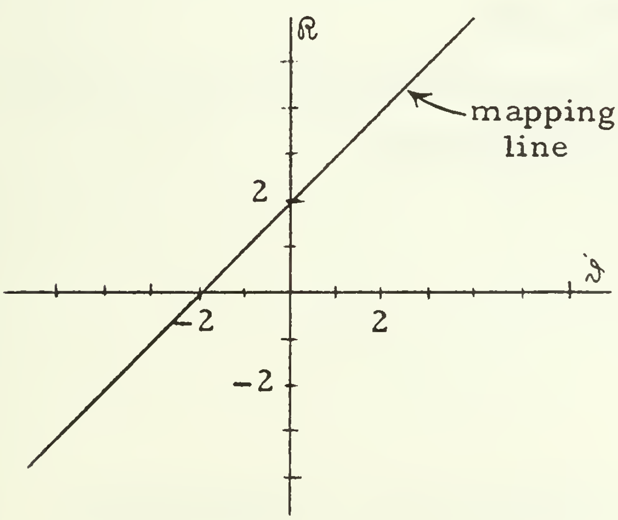


Fig. 4.

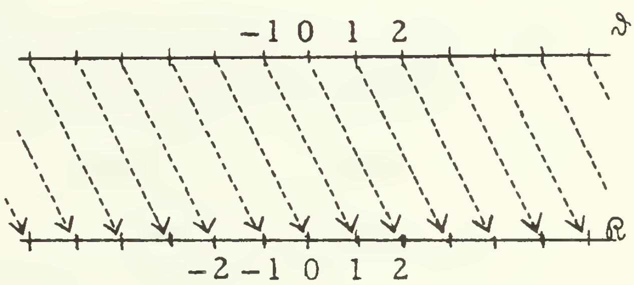
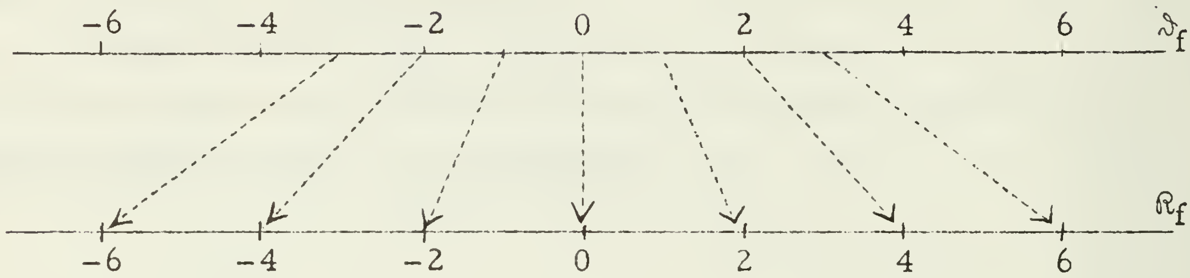
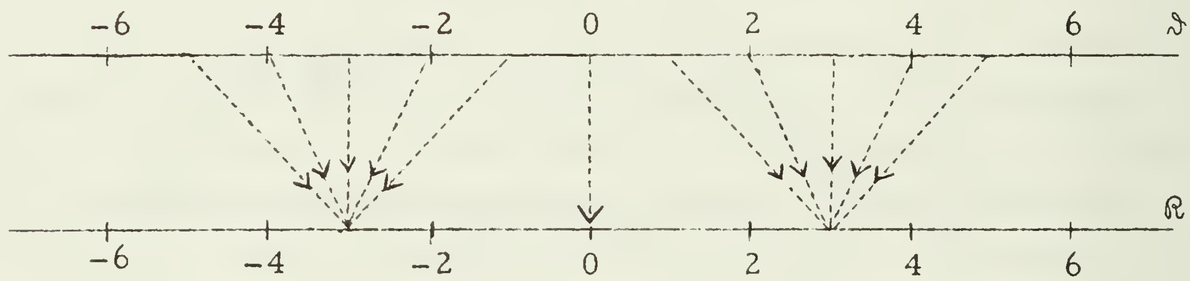


Fig. 5.

Here is a picture of a mapping determined by a function f. Give a brace-notation name for f.



The mapping shown below is determined by the function $\{(x, y): (y = -3 \text{ and } x < 0) \text{ or } (y = x \text{ and } x = 0) \text{ or } (y = 3 \text{ and } x > 0)\}$.



Draw a graph of this function.

Give a brace-notation name for the function which maps the set of real numbers on $\{2\}$. Make a picture like the ones above which shows this mapping. [What do we call such a mapping?]

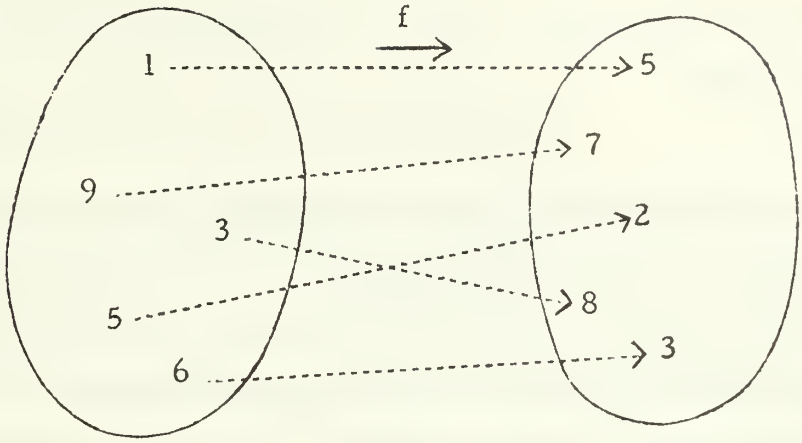
EXERCISES

A. Make pictures [like the illustrations immediately above] of the mappings determined by the listed functions.

1. $\{(x, y): x + y = 3\}$
2. $\{(x, y): y = x^2\}$
3. $\{(x, y), x \geq 0: y = \sqrt{x}\}$
4. $\{(x, y): y = |x|\}$
5. $\{(x, y): (x + y = -4 \text{ and } x \leq 0) \text{ or } (x + y = 4 \text{ and } x > 0)\}$

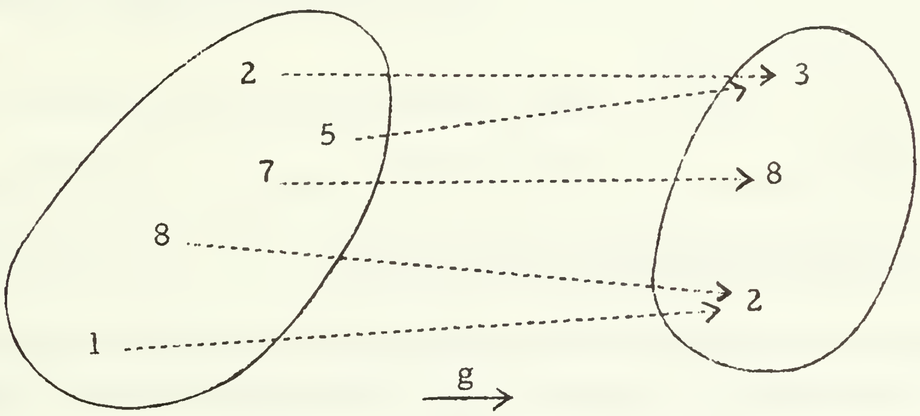
B. Each of the following exercises describes a mapping. Fill in the blanks.

1.



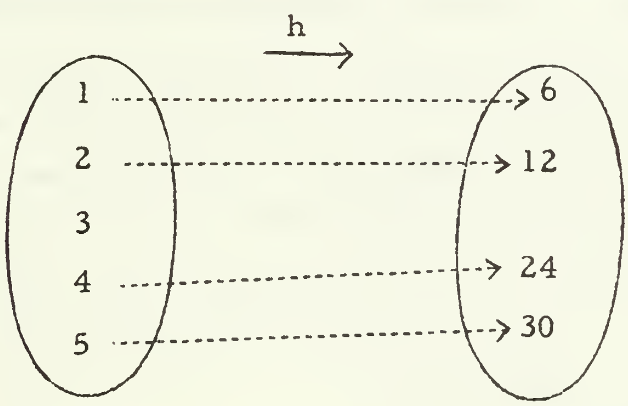
(a) f(1) = ____ (b) f(3) = ____ (c) f(____) = 3

2.



(a) g(7) = ____ (b) g(2) = ____ (c) g(____) = 2

3.



(a) h(2) = ____ (b) h(____) = 24 (c) h(3) = ____

4. f is the function which maps each real number on its opposite.
   (a) $f(7) = \_\_\_\_$ (b) $f(-\pi) = \_\_\_\_$ (c) $f(\_\_\_\_) = -\pi$

5. g is the function which maps each <u>positive</u> real number on its absolute value.
   (a) $g(9) = \_\_\_\_$ (b) $g(-2) = \_\_\_\_$ (c) $g(\_\_\_\_) = \sqrt{2}$

6. h is the function which maps each <u>integer</u> on its double.
   (a) $h(1) = \_\_\_\_$ (b) $h(\_\_\_\_) = 7$ (c) $h(7) = \_\_\_\_$
   (d) $h(3.5) = \_\_\_\_$ (e) $h(-5) = \_\_\_\_$ (f) $h(\_\_\_\_) = 0$

7. m maps each real number on its average with 0.
   (a) $m(5) = \_\_\_\_$ (b) $m(-7) = \_\_\_\_$ (c) $m(\_\_\_\_) = 0$
   (d) $m(\_\_\_\_) = 9.3$ (e) $m(\_\_\_\_) = -6.1$ (f) $m(\_\_\_\_) = \frac{2}{3}$

8. G maps each real number on the greatest integer less than or equal to the real number.
   (a) $G(3.7) = \_\_\_\_$ (b) $G(3.4) = \_\_\_\_$ (c) $G(-3.4) = \_\_\_\_$
   (d) $G(\frac{1}{2}) = \_\_\_\_$ (e) $G(-\frac{1}{2}) = \_\_\_\_$ (f) $G(\_\_\_\_) = 8$

WAYS OF REFERRING TO FUNCTIONS

Mathematicians have been working with functions for at least 300 years, and during this period people developed different notions of what functions are as well as different ways of talking about them. One of the more recent ways of regarding functions is to see them as sets of ordered pairs, and this is the point of view we have adopted in this course. We shall, of course, maintain our point of view throughout the course, but we shall occasionally use other ways of writing about functions just to prepare you to understand these other ways when you come upon them in books or in talking with other students. It turns out that the various concepts of functions and the various ways of referring to them can be interpreted to make sense from our point of view.

For example, people sometimes use phrases like:

(1) the function x^2
(2) the function $y = x^2$
(3) the function $f(x)$ where $f(x) = x^2$

These can all be interpreted as names for $\{(x, y): y = x^2\}$.

Sometimes people refer to this function by saying that it is

(4) the function defined by the expression 'x^2', or

(5) the function defined by the equation '$y = x^2$', or

(6) the function f defined by: $f(x) = x^2$.

This last is very much like the way we described functions in Exercises 10-13 of Part B on page 5-59. The difference is that in (6) the domain of the function is not specified. In accord with a common convention, when a function is described by a phrase like (6), its domain is understood to be the set of those real numbers for which the right side of the equation has values. So, for example, the domain of the function described by (6) is understood to be the set of *all* real numbers. Also, the domain of the function described by:

$$\text{the function g defined by: } g(x) = \frac{2}{1 - x}$$

is understood to be the set of real numbers different from 1. *We shall adopt this convention.* [Similar conventions are used by people who employ the other methods of referring to functions.]

Sometimes we talk about "places" where a function is defined, or where it isn't. Consider the function h where $h = \{(x, y): xy = 1\}$. Since 2 is an argument of h, but 0 is not, we say that h is defined at 2 but not at 0. [This corresponds to saying that '$h(2)$' names a member of the range of h but that '$h(0)$' is nonsense.] At which numbers are the functions described below not defined?

$$f_1(x) = \frac{1}{x - 2} \qquad f_2(x) = \frac{1}{x^2 - 1}$$

$$f_3 = \{(x, y): y(x - 1)(x + 2) = 1\}$$

$$f_4 = \{(x, y), \cancel{h}\, y \geq 0: y^2 = x\}$$

$$f_5(x) = \sqrt{x} \qquad f_6(x) = \sqrt{-x}$$

$$f_7 = \{(x, y), x \geq 0: y = x^2\}$$

EXERCISES

A. Each exercise refers to a function. Use brace-notation to name the function.

Sample 1. $q(x) = \frac{5}{2 - x}$

Solution. This refers to the function

$$\{(x, y),\ x \neq 2:\ y = \frac{5}{2 - x}\},$$

and introduces the name 'q' for this function.
[Another brace-notation name for q is: $\{(x, y):\ y(2 - x) = 5\}$]

1. the function $y = 5 - x$

2. the function $1 + 2x^2$

3. the function $y = 3x + 4$, for $x > 0$ [This means that the domain is the set of positive numbers.]

4. $f(x) = 1 + \sqrt{x}$

Sample 2. $f(x) = \begin{cases} 2, & \text{for } x \geq 0 \\ -2, & \text{for } x < 0 \end{cases}$

Solution. $\{(x, y):\ (y = 2 \text{ and } x \geq 0) \text{ or } (y = -2 \text{ and } x < 0)\}$

[An alternative is:
$\{(x, y):\ (\text{if } x \geq 0 \text{ then } y = 2) \text{ and } (\text{if } x < 0 \text{ then } y = -2)\}$]

5. $t(x) = \begin{cases} x, & \text{for } x \geq 1 \\ 2 - x, & \text{for } x < 1 \end{cases}$

6. $f(x) = 3$, for $x > 3$
$= x$, for $-3 \leq x \leq 3$
$= -3$, for $x < -3$

7. $g(x) = x + 3$, for $-2 \leq x \leq 2$

8. $d(x) = \begin{cases} -1, & \text{for } x \text{ a rational number} \\ 1, & \text{for } x \text{ an irrational number} \end{cases}$

B. Graph the ten functions given in Part A.

[Supplementary exercises are in Parts L and M, pages 5-254 to 5-255.]

EXPLORATION EXERCISES

Fill the blanks.

Sample. $F = \{(-1, 5), (1, 9), (2, 6), (-3, 7), (-4, 8)\}$
$G = \{(6, 14), (5, 16), (9, 18), (8, 20), (7, 22)\}$
(a) $G(F(1)) =$ _______ (b) $F(G(9)) =$ _______

Solution. (a) $G(F(1)) = G(9) = 18$
(b) $G(9) = 18$, but '$F(18)$' is nonsense; hence, so is '$F(G(9))$'.

1. $F_1 = \{(1, 2), (2, 4), (3, 7), (4, 10)\}$
   $F_2 = \{(2, 12), (7, 2), (10, 3), (4, 12)\}$

   (a) $F_1(2) =$ ____ (b) $F_2(F_1(2)) =$ ____ (c) $F_1(F_2(7)) =$ ____
   (d) $F_2(F_2(7)) =$ ____ (e) $F_1(F_2(4)) =$ ____ (f) $F_2(F_1(4)) =$ ____
   (g) $F_1(F_1(3)) =$ ____ (h) $F_1(F_2(F_1(3))) =$ ____

2. $f = \{(5, -3), (7, -3), (-2, -3), (1, -3)\}$
   $g = \{(4, 5), (2, 7), (-3, 1), (6, 6), (5, 4)\}$

   (a) $f(5) =$ ____ (b) $f(1) =$ ____ (c) $g(f(5)) =$ ____
   (d) $g(f(-2)) =$ ____ (e) $f(g(-3)) =$ ____ (f) $g(f(-3)) =$ ____
   (g) $g(g(5)) =$ ____ (h) $g(g(g(g(5)))) =$ ____
   (i) $g(g(g(6))) =$ ____ (j) $f(f(5)) =$ ____

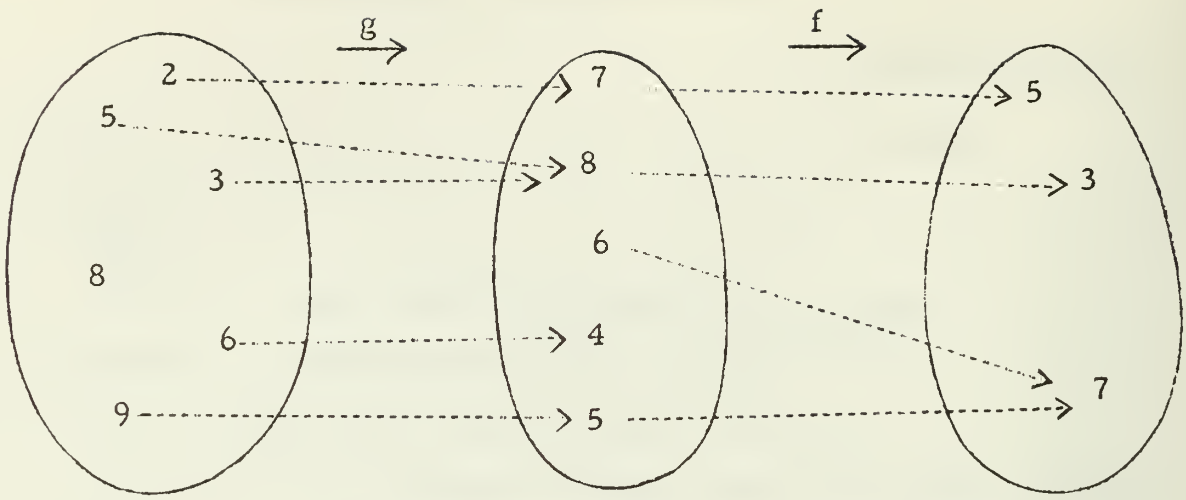
3. $b = \{(x, y):\ y = 5x + 2\}$ and $g = \{(x, y):\ y = 2x - 5\}$

   (a) $b(2) =$ ____ (b) $b(b(3)) =$ ____ (c) $g(-1) =$ ____
   (d) $g(g(g(5))) =$ ____ (e) $b(g(3)) =$ ____ (f) $g(b(3)) =$ ____
   (g) $b(b(g(b(0)))) =$ ____ (h) $b(g($____$)) = 4$

4. $G(x) = 7x - 2$ and $H(x) = \frac{x + 2}{7}$

   (a) $G(3) =$ ____ (b) $H(19) =$ ____ (c) $G(5) =$ ____
   (d) $H(G(5)) =$ ____ (e) $H(G(100)) =$ ____ (f) $G(H(1002) =$ ____
   (g) $\mathfrak{D}_G =$ ____________ (h) $\mathfrak{R}_G =$ ____________
   (i) $\mathfrak{D}_H =$ ____________ (j) $\mathfrak{R}_H =$ ____________

5.



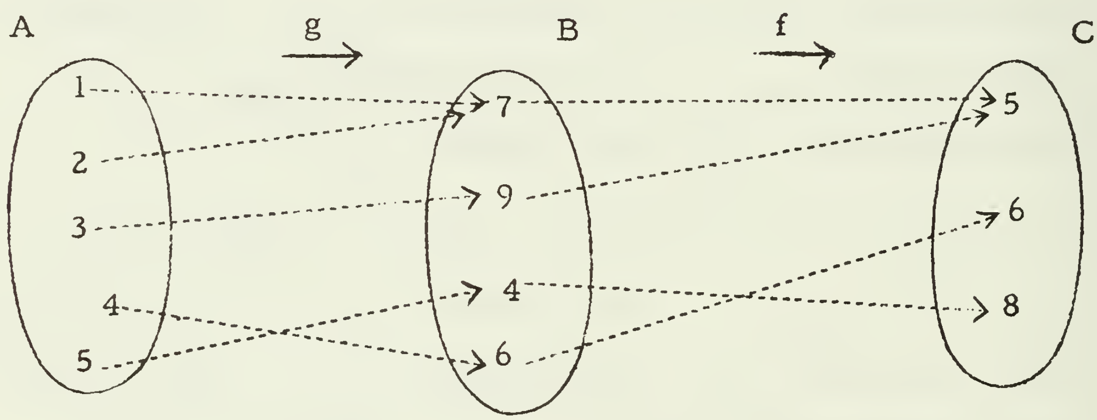
(a) $g(5) =$ ____ (b) $g($____$) = 4$ (c) $f(5) =$ ____

(d) $g(8) =$ ____ (e) $f(8) =$ ____ (f) $f($____$) = 1$

(g) $f(g(2)) =$ ____ (h) $f(g(3)) =$ ____ (i) $f(g($____$)) = 3$

(j) $f(g(7)) =$ ____ (k) $f(g($____$)) = 7$ (ℓ) $f(g($____$)) = 1$

6. Here is a picture which shows a mapping g of A on B, and a mapping f of B on C.



(a) $g = \{$____, ____, ____, ____, ____$\}$

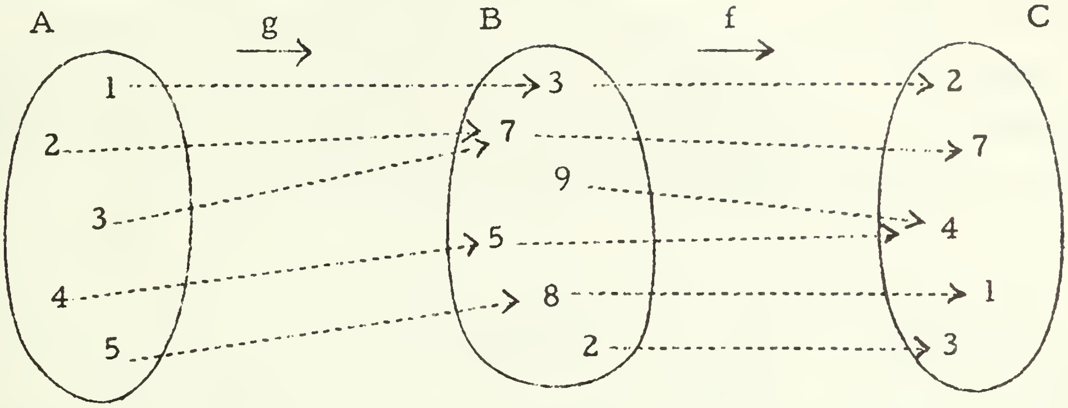
(b) $f = \{$____, ____, ____, ____$\}$

(c) $\mathcal{R}_g =$ ____________, $\mathcal{D}_f =$ ____________

(d) The picture suggests a mapping h of A on C.

$h = \{$____, ____, ____, ____, ____$\}$

7. Here is a picture which shows a mapping g of A on a subset of B, and a mapping f of B on C.



(a) g = {____, ____, ____, ____, ____}

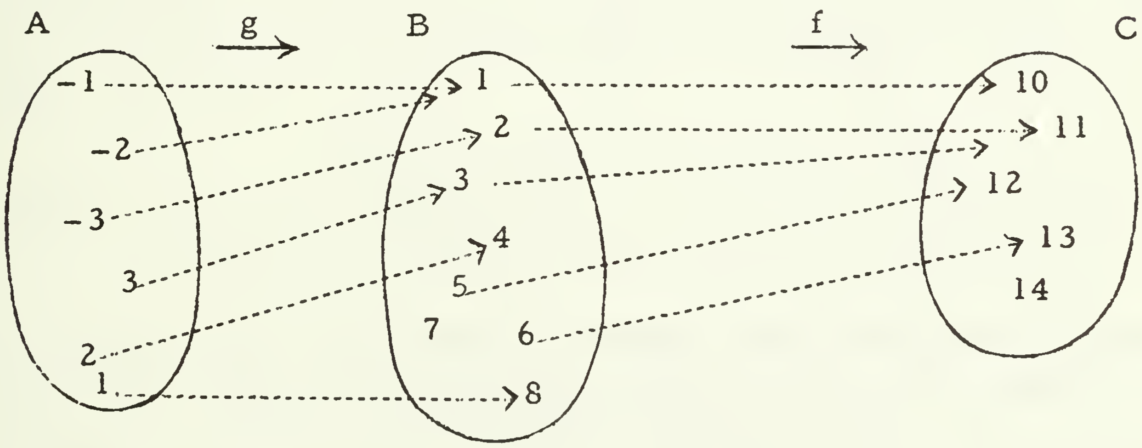
(b) f = {____, ____, ____, ____, ____, ____}

(c) $\mathcal{R}_g$ = ____________, $\mathcal{D}_f$ = ____________

(d) The picture suggests a mapping h of A on a subset of C.

h = {____, ____, ____, ____, ____}

8. Here is another picture of two mappings f and g.



(a) g = {____, ____, ____, ____, ____, ____}

(b) f = {____, ____, ____, ____, ____}

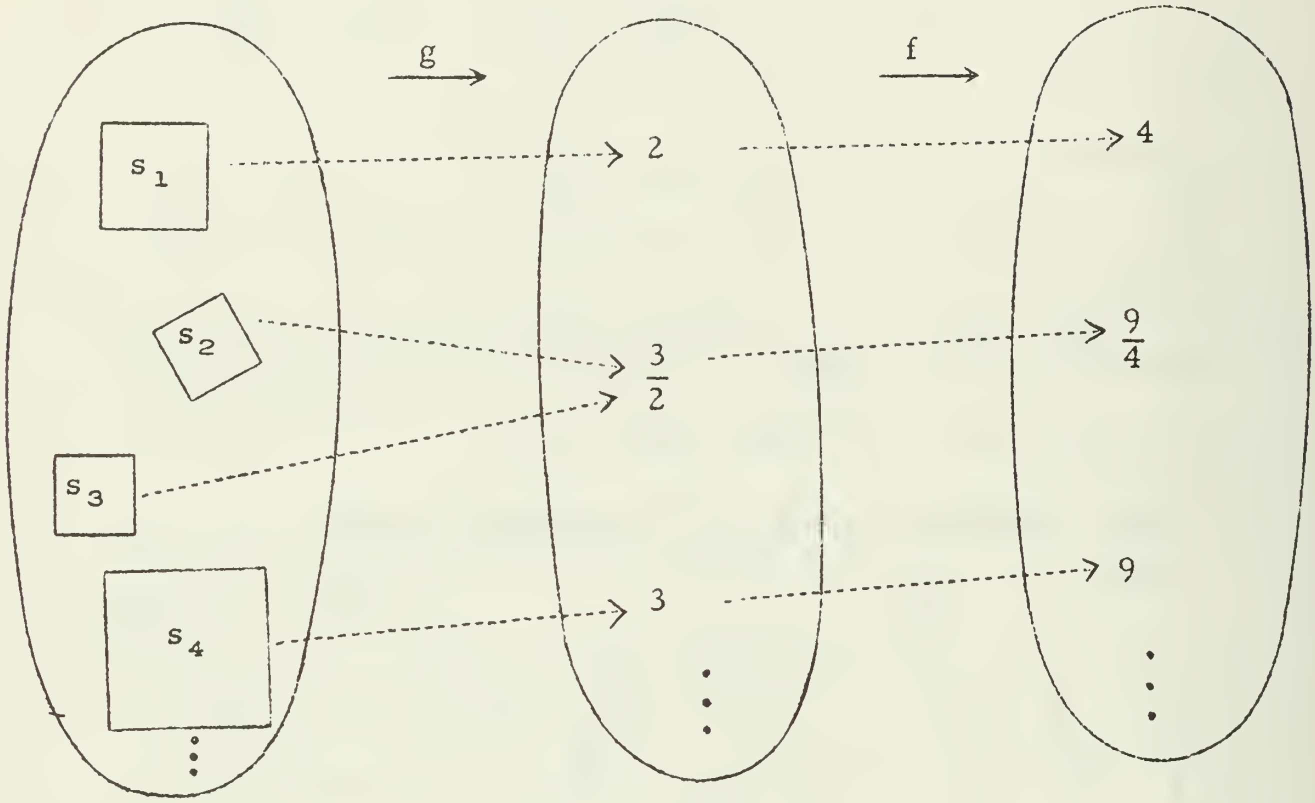
(c) $\mathcal{R}_g$ = ____________, $\mathcal{D}_f$ = ____________

(d) This picture does not suggest a mapping of A on a subset of C because although g maps 2 on 4 and 1 on 8, neither 4 nor 8 is in the domain of f. However, the picture does suggest a mapping, h, of the subset {-1, -2, -3, 3} of A on a subset of C.

h = {____, ____, ____, ____}

COMPOSING FUNCTIONS

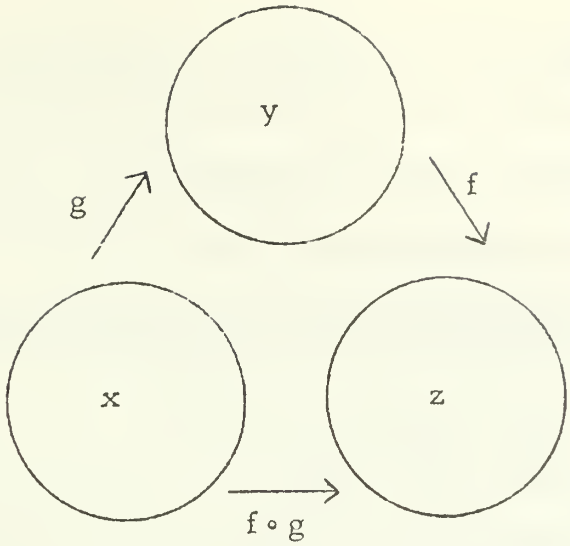
Suppose g is the function which maps each square on its side-measure. [Sometimes we call g 'the side-measure of a square'.] And, suppose f is the function $\{(x, y) \in N \times N: y = x^2\}$ which maps each number of arithmetic on the square of itself. Does the picture suggest a mapping of squares on their area-measures?



Do you see that, for each square s, f(g(s)) is the area-measure of s?

What we have just done is to describe what someone does when he computes the area-measure of a square. First, he uses g to find the side-measure. [That is, he measures the side of the square.] Then, he uses f to find the area-measure. [That is, he multiplies the side-measure by itself.]

This is also an example of a way of combining two functions to get a third function. This <u>operation</u> on functions is called <u>composition</u>. In the example above, the third function is the area-measure of a square. It was obtained by composing f with g. We call the function obtained by composing f with g: $f \circ g$. [Read '$f \circ g$' as 'f composed with g' or 'f of g' or 'f circle g'.]



If g maps x on y, and f maps y on z, then $f \circ g$ maps x on z.

Let's take another example. Consider the functions p and q where

$$p = \{(0, 4), (1, 9), (2, -9), (3, 25), (4, -16)\}$$

and

$$q = \{(x, y), x \geq 0: y = \sqrt{x}\}.$$

What is $q \circ p$?

The function p maps 0 on 4, and q maps 4 on 2. So, $q \circ p$ is a function which maps 0 on 2. Similarly, $q \circ p$ maps 1 on 3. However, although p maps 2 on -9, -9 does not belong to the domain of q. So, $q \circ p$ is not defined at 2. Similarly, $q \circ p$ is not defined at 4. But, $[q \circ p](3) = 5$. Thus,

$$q \circ p = \{(0, 2), (1, 3), (3, 5)\}$$

What is the function obtained by composing p with q? That is, what is $p \circ q$? To find the ordered pairs in $p \circ q$, we could begin by searching among the ordered pairs in q for those whose second components belong to $\mathcal{D}_p$. One such ordered pair in q is (0, 0). The others are (1, 1), (4, 2), (9, 3), and (16, 4). Now, since

$$\begin{array}{ccccc} & \xrightarrow{q} & & \xrightarrow{p} & \\ 0 & \dashrightarrow & 0 & \dashrightarrow & 4 \\ 1 & \dashrightarrow & 1 & \dashrightarrow & 9 \\ 4 & \dashrightarrow & 2 & \dashrightarrow & -9 \\ 9 & \dashrightarrow & 3 & \dashrightarrow & 25 \\ 16 & \dashrightarrow & 4 & \dashrightarrow & -16 \end{array}$$

it follows that

$$p \circ q = \{(0, 4), (1, 9), (4, -9), (9, 25), (16, -16)\}.$$

Here is a definition of the operation of composition:

> For each function f, for each function g,
>
> $f \circ g$ is the function such that
>
> (1) $[f \circ g](x) = f(g(x))$, for each $x \in \mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$,
>
> and
>
> (2) $\mathcal{D}_{f \circ g} = \{x \in \mathcal{D}_g: \ g(x) \in \mathcal{D}_f\}$.

[Notice that $\mathcal{D}_{f \circ g} \subseteq \mathcal{D}_g$. Under what condition does $\mathcal{D}_{f \circ g} = \mathcal{D}_g$?]

EXERCISES

A. Fill in the blanks.

1. $g = \{(6, 2), (9, 4), (12, -1), (5, 3)\}$
   $f = \{(x, y): \ y = x^2\}$
   $f \circ g = \{$ ____________________ $\}$

2. $a = \{$(John, 7), (Bill, 5), (Emma, 8)$\}$
   $b = \{(x, y), \ x > 6: \ y = 3x + 1\}$
   $b \circ a = \{$ ____________________ $\}$

3. $s = \{(2, 1), (3, 2), (5, 1), (6, 0), (8, 2)\}$
   $t = \{(0, 2), (1, 3), (2, 6), (3, 8), (4, 1), (5, 0)\}$
   (a) $t \circ s = \{$ ____________________ $\}$
   (b) $s \circ t = \{$ ____________________ $\}$

4. $m = \{(0, 4), (1, 2), (2, 3), (3, 8), (4, 10)\}$
   $n = \{(0, 1), (2, 3), (4, 5), (6, 7), (8, 9), (10, 11)\}$
   (a) $n \circ m = \{$ ____________________ $\}$
   (b) $m \circ n = \{$ ____________________ $\}$

5. s is the function which maps each equilateral triangle on its side-measure, and t is the tripling function for numbers of arithmetic.

   $t \circ s$ is ____________________________

<u>Sample</u>. $g = \{(x, y): y = 2x\}$
$f = \{(x, y): y = x^3\}$

(a) $f \circ g =$ ________

(b) $g \circ f =$ ________

<u>Solution</u>. Since $\mathcal{R}_g \subseteq \mathcal{D}_f$, the domain of $f \circ g$ is $\mathcal{D}_g$ the set of real numbers. For each real number x,

$$[f \circ g](x) = f(g(x)) = f(2x) = (2x)^3 = 8x^3.$$

So, $f \circ g = \{(x, y): y = 8x^3\}$.

Similarly, since $\mathcal{R}_f \subseteq \mathcal{D}_g$, the domain of $g \circ f$ is $\mathcal{D}_f$, the set of real numbers. For each real number x,

$$[g \circ f](x) = g(f(x)) = g(x^3) = 2x^3.$$

So, $g \circ f = \{(x, y): y = 2x^3\}$.

6. $k = \{(x, y): y = 3x\}$
$q = \{(x, y): y = x^4\}$

(a) $q \circ k =$ ________________

(b) $k \circ q =$ ________________

7. $c = \{(x, y): y = x - 5\}$
$d = \{(x, y): y = |x|\}$

(a) $d \circ c =$ ________________

(b) $c \circ d =$ ________________

8. $g(x) = x + 3$
$f(x) = x^2$

(a) $[f \circ g](x) =$ ____________

(b) $[g \circ f](x) =$ ____________

9. $r(x) = x - 3$
$s(x) = x + 3$

(a) $[s \circ r](x) =$ ____________

(b) $[r \circ s](x) =$ ____________

10. $p(x) = x - 3$, $q(x) = x^2$

(a) $[q \circ p](3) =$ _____ (b) $[q \circ p](7) =$ _____ (c) $[q \circ p](2) =$ _____

(d) $[p \circ q](2) =$ _____ (e) $[p \circ q]($_____$) = 22$ (f) $[q \circ p]($_____$) = -4$

11. $s(m) = 3m + 5$, $t(m) = (m - 2)^2$

(a) $[t \circ s](1) =$ _____ (b) $[s \circ t](1) =$ _____ (c) $[t \circ s](-1) =$ _____

(d) $[s \circ t](-1) =$ _____ (e) $[t \circ s]($____$) = 81$ (f) $[s \circ t]($____$) = 5$

12. $k(t) = t^2 + 1$

$j(t) = 2t - 1$

(a) $[j \circ k](t) =$ ______________

(b) $[k \circ j](t) =$ ______________

13. $k(t) = t^2 + 1$

$$j(t) = \begin{cases} 2t - 1, & \text{for } t \geq 1 \\ 7 - t, & \text{for } t < 1 \end{cases}$$

(a) $[j \circ k](t) =$ ______________

(b) $[k \circ j](t) =$ ______________

14. $g(x) = 2x - 3$

$f(x) = 7$

(a) $[f \circ g](x) =$ ______________

(b) $[g \circ f](x) =$ ______________

☆15. $g = \{(x, y), x > 0: y = 2x\}$

$f = \{(x, y), x > 0: y = -\sqrt{x}\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

16. $g = \{(4, 3), (5, 7), (6, -1)\}$

$f = \{(3, 4), (7, 5), (-1, 6)\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

17. $g = \{(x, y): y = 2x + 5\}$

$f = \{(x, y): y = \frac{x - 5}{2}\}$

(a) $f \circ g =$ ______________

(b) $g \circ f =$ ______________

18. $g = \{(x, y): y = 3x + 7\}$

(a) $g \circ g =$ ______________

(b) $g \circ [g \circ g] =$ ______________

19. $g = \{(x, y): 2x + y = 3\}$

(a) $g \circ g =$ ______________

(b) $[g \circ g] \circ g =$ ______________

20. $g(x) =$ the side-measure of square x, $\mathscr{D}_g =$ the set of all squares.

$f(x) = 4x$, $\mathscr{D}_f =$ the set of numbers of arithmetic.

$[f \circ g](x) =$ ______________, $\mathscr{D}_{f \circ g} =$ ______________

☆21. $g(x) =$ the husband or wife of x, $\mathscr{D}_g =$ the set of married people.

$f(x) =$ the father of x, $\mathscr{D}_f =$ the set of married people.

(a) $[f \circ g](x) =$ ______________, $\mathscr{D}_{f \circ g} =$ ______________

(b) $[g \circ f](x) =$ ______________, $\mathscr{D}_{g \circ f} =$ ______________

[Supplementary exercises are in Part N, pages 5-255 through 5-257.]

B. Compose each of the functions f_1, f_2, f_3, and f_4 with g where

$$g = \{(1, 1), (2, 0), (3, 2)\},$$

and

$f_1 = \{(x, y): y = 2x\}$, $f_2 = \{(x, y), x \geq 0: y = 2x\}$,

$f_3 = \{(1, 2), (0, 0), (2, 4)\}$, $f_4 = \{(x, y): y = x^3 - 3x^2 + 4x\}$.

$f_1 \circ g =$ ________________ $f_2 \circ g =$ ________________

$f_3 \circ g =$ ________________ $f_4 \circ g =$ ________________

What must you know about a function f to predict that $f \circ g = f_1 \circ g$? How many such functions f are there? How many are there such that $\mathfrak{D}_f = \mathfrak{R}_g$?

C. 1. Show that the operation composition-of-functions is not commutative. That is, show that '$\forall_f \forall_g\ f \circ g = g \circ f$' is false.

2. Fill in the blanks.

$f(x) = 2x$, $g(x) = x + 5$, and $h(x) = x - 3$.

(a) $h(7) =$ ______ (b) $[f \circ g](4) =$ ______

(c) $[f \circ g](h(7)) =$ ______ (d) $[[f \circ g] \circ h](7) =$ ______

(e) $g(h(11)) =$ ______ (f) $[g \circ h](11) =$ ______

(g) $f([g \circ h](11)) =$ ______ (h) $[f \circ [g \circ h]](11) =$ ______

(i) $[[f \circ g] \circ h](9) =$ ______ (j) $[f \circ [g \circ h]](9) =$ ______

(k) $[f \circ g](x) =$ ______ (ℓ) $[[f \circ g] \circ h](x) =$ ______

(m) $[g \circ h](x) =$ ______ (n) $[f \circ [g \circ h]](x) =$ ______

3. The preceding exercise suggests that the operation composition-of-functions is associative, that is, that

$$\forall_f \forall_g \forall_h\ [f \circ g] \circ h = f \circ [g \circ h].$$

Let's try to prove this.

(a) Suppose x belongs to the domain of $[f \circ g] \circ h$. Then, by the definition on page 5-74, $x \in \mathfrak{D}_h$ and $h(x) \in \mathfrak{D}_{f \circ g}$. Also,

$$[[f \circ g] \circ h](x) = [f \circ g](h(x)).$$

Since $h(x) \in \mathfrak{D}_{f \circ g}$, it follows from the same definition that

$$[f \circ g](h(x)) = \underline{\qquad\qquad\qquad}.$$

So, if x belongs to the domain of $[f \circ g] \circ h$,

$$[[f \circ g] \circ h](x) = \underline{\qquad\qquad\qquad}.$$

(b) Now, suppose x belongs to the domain of $f \circ [g \circ h]$. Again, by definition, $x \in \mathfrak{D}_{g \circ h}$ and $[g \circ h](x) \in \mathfrak{D}_f$. Also,

$$[f \circ [g \circ h]](x) = f([g \circ h](x)).$$

Since $x \in \mathfrak{D}_{g \circ h}$, it follows from the definition that

$$[g \circ h](x) = \underline{\qquad\qquad\qquad}.$$

So, if x belongs to the domain of $f \circ [g \circ h]$,

$$[f \circ]g \circ h]](x) = \underline{\qquad\qquad\qquad}.$$

(c) In parts (a) and (b), you have shown that, for each x which belongs both to the domain of the function $[f \circ g] \circ h$ and to the domain of the function $f \circ [g \circ h]$, the functions have the same value. But, this does not yet prove that they are the same function. To do so, we must prove that the domain of $[f \circ g] \circ h$ is the domain of $f \circ [g \circ h]$. Let's see what it means to say that x belongs to the domain of $[f \circ g] \circ h$. By definition, this is the case if and only if

$$x \in \mathfrak{D}_h \text{ and } h(x) \in \mathfrak{D}_{f \circ g}.$$

Again, by definition, <u>this</u> is the case if and only if

$$(*) \qquad x \in \mathfrak{D}_h \text{ and } [h(x) \in \mathfrak{D}_g \text{ and } g(h(x)) \in \mathfrak{D}_f].$$

Now what does it mean to say that x belongs to the domain of $f \circ [g \circ h]$? [Use the definition to show that 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*). This will prove that $[f \circ g] \circ h$ and $f \circ [g \circ h]$ have the same domain.] So, since $[f \circ g] \circ h$ and $f \circ [g \circ h]$ have the same domain, and since, for each x in this domain, $[[f \circ g] \circ h](x) = [f \circ [g \circ h]](x)$, it follows that $[f \circ g] \circ h = f \circ [g \circ h]$.

THE INVERSE OF A FUNCTION

You will recall from earlier units that operations such as adding 2 are sets of ordered pairs. Other examples of operations are multiplying by -1, absolute valuing, and square rooting. Since an operation is a set of ordered pairs, an operation is a relation. More particularly, an operation is a function. [In fact, you may recall reading (on page 1-107) that an operation is a set of ordered pairs no two of which have the same first component.] You also learned that an operation has an inverse just if the set of ordered pairs obtained by reversing the components of each of its ordered pairs is an operation [that is, just if the converse is an operation]. In general, a function is said to have an inverse if and only if the converse of the function is also a function. And, in this case, the converse is called <u>the inverse of the given function</u>.

(1) Give three ordered pairs which belong to adding -5.
(2) Give three pairs which belong to the inverse of adding -5.
(3) Is the converse of multiplying by 2 a function?
(4) Does multiplying by 2 have an inverse?
(5) Does multiplying by 0 have a converse?
(6) Does multiplying by 0 have an inverse?
(7) Does absolute valuing have an inverse?
(8) Does absolute valuing have a converse?
(9) Does squaring have an inverse?
(10) Does square rooting have an inverse?

How many inverses can a function have? If a function has an inverse, what is <u>its</u> inverse? If g is the inverse of a function f, what are $\mathcal{D}_g$ and $\mathcal{R}_g$? Give an example of a function which is its own inverse. If g is the inverse of a function f, what can you say about the relation $g \cup f$?

EXERCISES

<u>A</u>. For each function which has an inverse, describe its inverse.

<u>Sample 1</u>. $G = \{(3, 1), (2, -5), (4, 3)\}$

<u>Solution</u>. The converse of G is $\{(1, 3), (-5, 2), (3, 4)\}$. This is a function. So, G has this function as its inverse.

Sample 2. $f(x) = 3x - 5$, $\mathcal{D}_f$ = the set of real numbers

Solution. Let g be the converse of f. Since

$$f = \{(x, y):\ y = 3x - 5\},$$

it follows from the definition of converse that

$$g = \{(y, x):\ y = 3x - 5\}. \quad \text{[Explain.]}$$

Is g a function? It is just if there are not two ordered pairs in g with the same first component. Suppose someone tells you the first component of an ordered pair in g. Can you tell him the second component of this ordered pair? Referring back to the brace-notation name for g, it is easy to see that given a first component p, the corresponding second components are just those numbers x such that $p = 3x - 5$. But, $p = 3x - 5$ if and only if $x = \frac{p + 5}{3}$. So, for each first component p there is just one second component q, the number $\frac{p + 5}{3}$. So, g is a function. In fact, we have shown that

$$g = \{(p, q):\ q = \frac{p + 5}{3}\}$$

$$= \{(x, y):\ y = \frac{x + 5}{3}\}.$$

Do you see a quick way of getting this last brace-notation name for g from the name for f?

Another description of g is:

$$g(x) = \frac{x + 5}{3}, \quad \mathcal{D}_g = \text{the set of real numbers}$$

Sample 3. $F = \{(x, y):\ y = x^2\}$

Solution. Suppose G is the converse of F. Then

$$G = \{(x, y):\ x = y^2\}$$

F has an inverse if and only if G is a function. But, G is not a function because each positive real number is the square of two real numbers. For example, both (4, 2) and (4, −2) belong to G.

1. $f = \{(-5, 6), (-6, 7), (-7, -10), (1/2, 1/3)\}$

2. $F = \{(1, 5), (2, 9), (4, 9)\}$

3. $g = \{(x, y):\ y = 5\}$

4. $h = \{(1, 3), (3, 1), (6, 2), (2, 6), (0, 0)\}$
5. $f(x) = 4x + 7$, $\mathscr{D}_f$ = the set of real numbers
6. $h(x) = 3 - 2x$, $\mathscr{D}_h$ = the set of real numbers
7. $G(x) = \frac{5x + 1}{9}$, $\mathscr{D}_G = \{x: x \geq 0\}$
8. $F(x) = |x| - 3$, $\mathscr{D}_F$ = the set of real numbers
9. $G(r) = -2r + 10$, $\mathscr{D}_G$ = the set of real numbers
10. $H(t) = t^2 + 5$, $\mathscr{D}_H = \{x: x \geq -1\}$ [Is there a subset of H which has an inverse? Is there more than one such subset of H?]

*

11. (a) A relation is a function if and only if no two ordered pairs in the relation have the same ________ component.
    (b) A function has an inverse if and only if no two ordered pairs in the function have the same _________ component.

12. (a) If g is the inverse of a function f then

$$\mathscr{D}_g = \_\_\_\_\_\_ \text{ and } \mathscr{R}_g = \_\_\_\_\_\_.$$

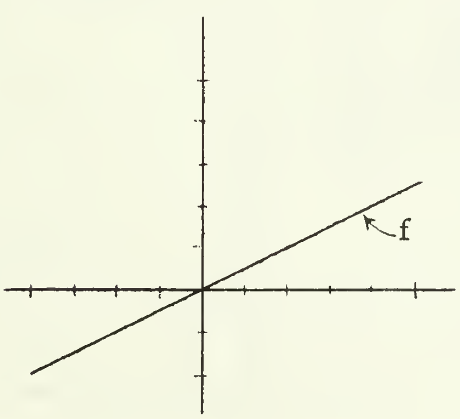
    (b) If g is the inverse of a function f then,

for each $k \in \mathscr{D}_f$, $[g \circ f](k) =$ ________.

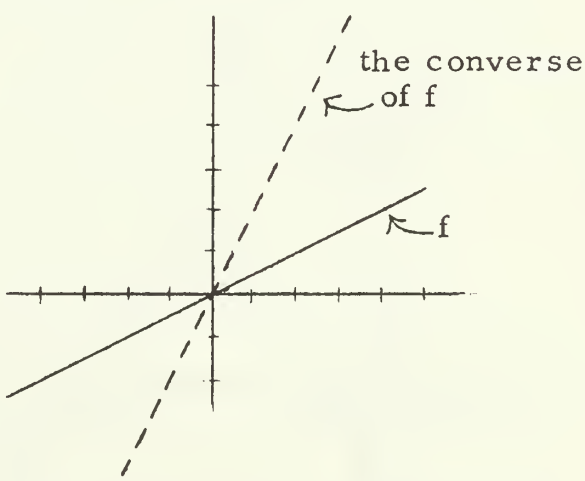
and, for each $k \in \mathscr{D}_g$, $[f \circ g](k) =$ ________.

B. Here are graphs of functions. Sketch the converse of each function, and tell whether the function has an inverse. [Try to predict which functions have inverses before you sketch the converses.]

Sample.

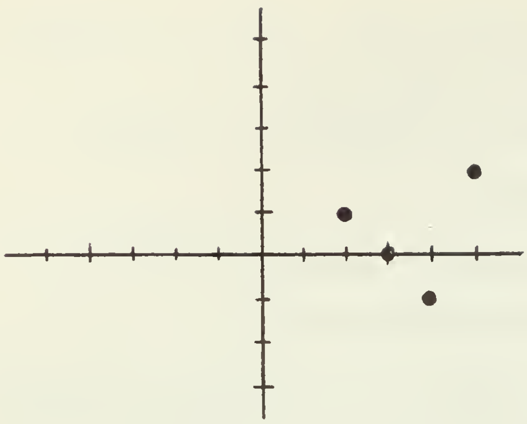


Solution.



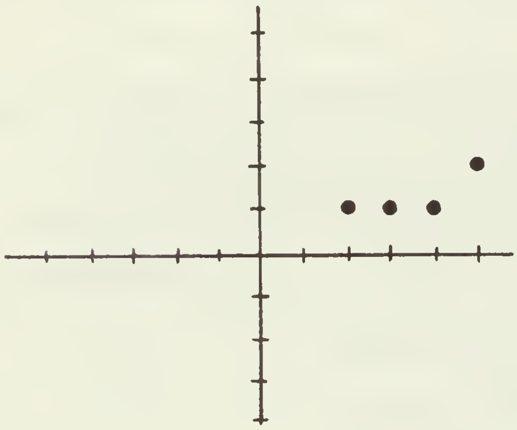
The converse of f is a function. So, f has an inverse.

1.

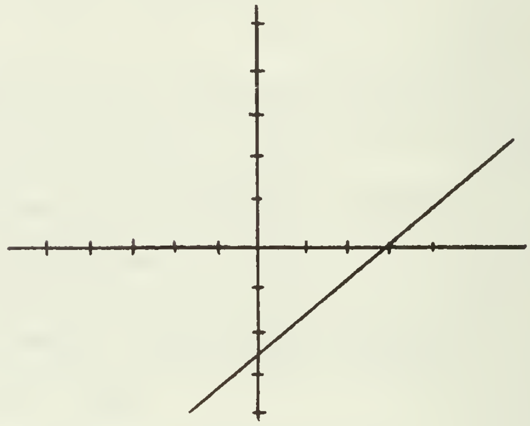


2.

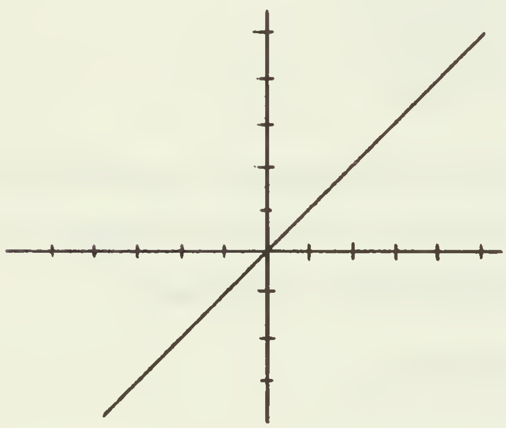
3.



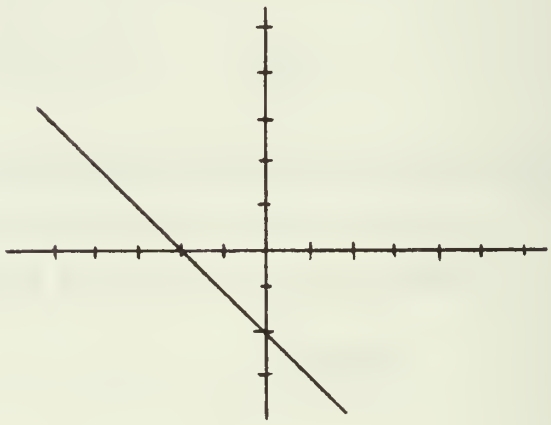
4.



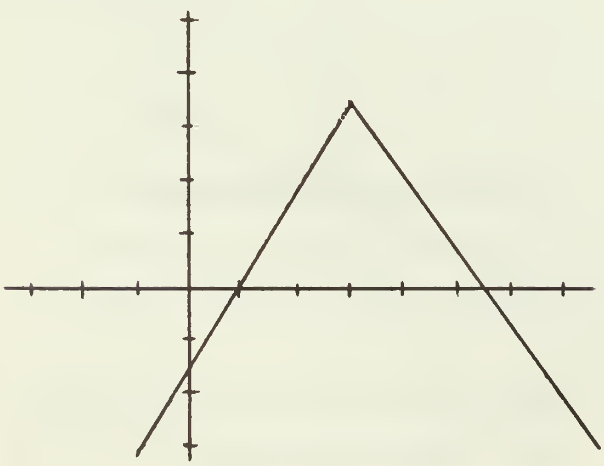
5.



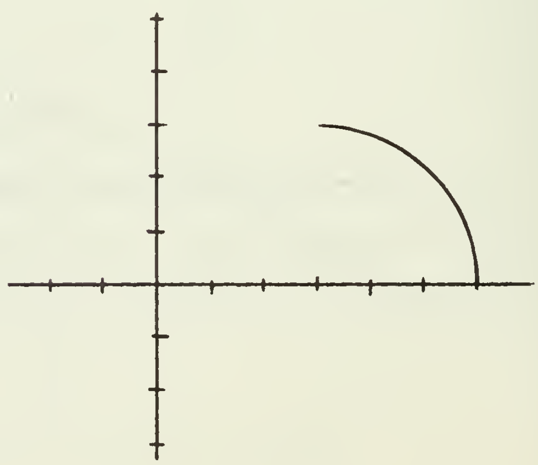
6.



7.



8.



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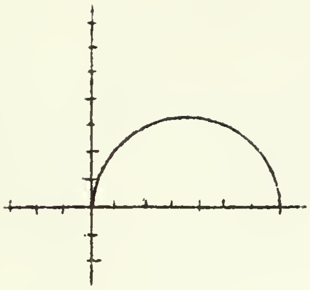
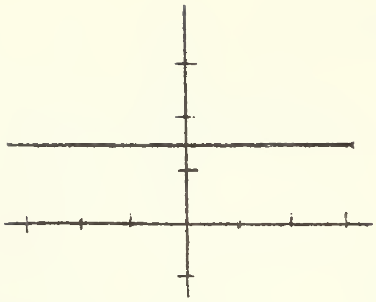


10.



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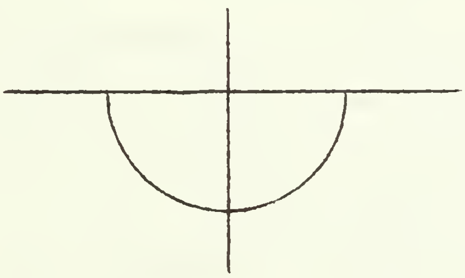
Functions such as the one pictured in Exercise 10 [which has the set of real numbers as its domain and whose ordered pairs all have the same second component] are called <u>constant functions</u>. Does a constant function have an inverse?

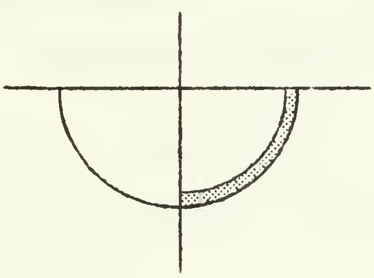
<u>C</u>. Here are pictures of functions which do not have inverses. However, it is the case that every function has subsets which have inverses. For each function pictured below and on page 5-84, choose a subset which has an inverse. Indicate your choice by marking the drawing.

<u>Sample.</u>

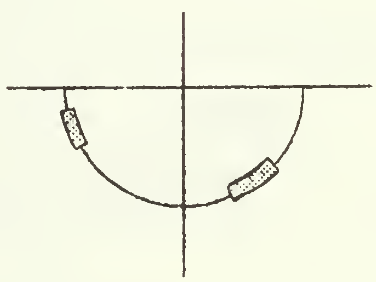


<u>Solution.</u> Here are just two of many possible answers.

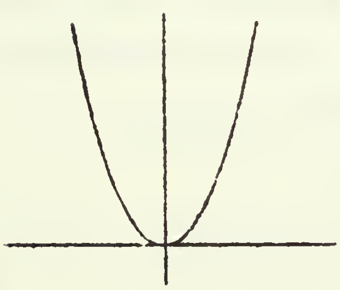
(I)



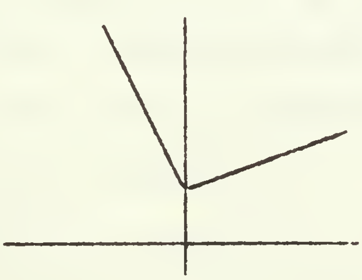
(II)



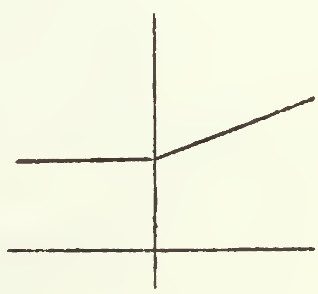
1.



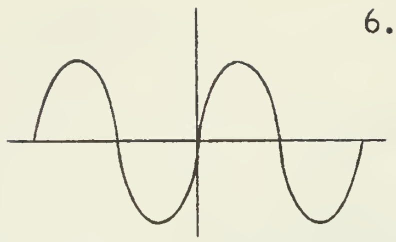
2.

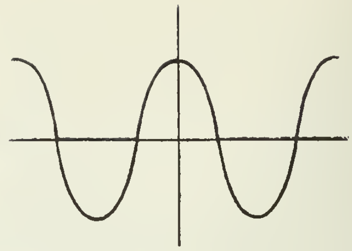


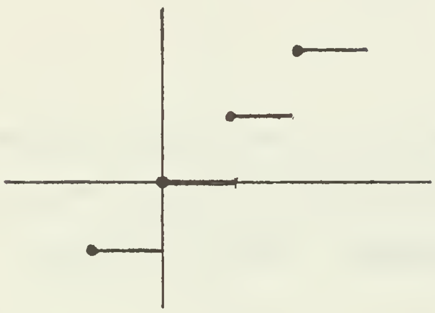
3.

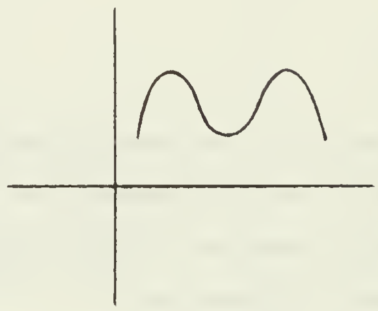


4. 

5. 

6. 

7. 

8. 

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9. Do you think that each relation has a subset which is a function whose domain is the domain of the relation? Do you think that each function has a subset which has an inverse whose domain is the range of the given function?

D. It is customary to use 'f^{-1}' as an abbreviation for 'the inverse of f'. [You can also read 'f^{-1}' as 'f inverse'.] Practice using this notation in simplifying the expressions given in the exercises. The exercises refer to the functions F_1, F_2, F_3, F_4, and F_5, where the domain of each is the set of real numbers and

$$F_1(x) = 2x + 3, \qquad F_2(x) = \frac{x}{3} - 1, \qquad F_3(x) = -4x$$

$$F_4(x) = x + 2, \qquad F_5(x) = -2x - 4$$

Notice that each of these functions has an inverse.

<u>Sample 1</u>. $F_3^{-1}(F_2(6))$

<u>Solution</u>. $F_2(6) = \frac{6}{3} - 1 = 1.$

$F_3^{-1}(1) = ?$

To simplify '$F_3^{-1}(1)$' we must find a number x such that $F_3(x) = 1$, that is, such that $-4x = 1$. Such a number is $-1/4$. Therefore, $F_3^{-1}(1) = -1/4$. So,

$$F_3^{-1}(F_2(6)) = -\frac{1}{4}.$$

[Note that: $y = F_3^{-1}(x)$, $(x, y) \in F_3^{-1}$, $(y, x) \in F_3$, and: $x = F_3(y)$, all say the same thing.]

Sample 2. $F_4(F_1(3a))$

Solution. $F_1(3a) = 2(3a) + 3 = 6a + 3,$

$F_4(6a + 3) = (6a + 3) + 2 = 6a + 5.$

So, $F_4(F_1(3a)) = 6a + 5.$

1. $F_3^{-1}(F_1(-2))$
2. $F_1^{-1}(F_2(-2))$
3. $F_2(-2) + F_1(-2)$
4. $F_1(-2) \cdot F_2(-2)$
5. $F_1(3) + F_2(4)$
6. $F_1(4) + F_3(3)$
7. $F_1(2r + 1)$
8. $F_5(1 - 3t)$
9. $F_2^{-1}(F_1^{-1}(-2))$
10. $F_1^{-1}(F_2^{-1}(-2))$
11. $F_1^{-1}(F_1(-2))$
12. $F_2(F_2^{-1}(-2))$
13. $F_3(F_2(4))$
14. $F_4(F_2^{-1}(0))$
15. $5 \cdot F(5)$
16. $5 \cdot F_5^{-1}(1/5)$
17. $F_2^{-1}(a)$
18. $F_5^{-1}(5b - 3)$
19. $F_4^{-1}(F_5(-5/2))$
20. $F_2^{-1}(F_2^{-1}(3))$
21. $F_4^{-1}(F_4(F_4^{-1}(7)))$
22. $F_3^{-1}(F_3(7a^2 + 2a - 8))$
23. $[F_1^{-1} \circ F_1](5)$
24. $[F_1 \circ F_1^{-1}](5)$
25. $[F_2^{-1} \circ F_2](843)$
26. $[F_2 \circ F_2^{-1}](927)$

[Supplementary exercises are in Parts O and P, pages 5-257ff.]

EXPLORATION EXERCISES

In each of the following exercises you are given functions g and h. Your job is to find, if possible, a function f such that $h = f \circ g$.

Sample 1. $g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$

$f =$ ____________________

$h = \{(3, 9), (5, 12), (8, 7), (7, 9)\}$

Solution. We are looking for a function f such that $f(6) = 9$, $f(9) = 12$, and $f(4) = 7$. So, such a function must contain the ordered pairs (6, 9), (9, 12), and (4, 7). One such function is f where $f = \{(6, 9), (9, 12), (4, 7)\}$. To see that this works, we find the ordered pairs in $f \circ g$:

$f(g(3)) = f(6) = 9$ $\qquad$ $f(g(5)) = f(9) = 12$

$f(g(8)) = f(4) = 7$ $\qquad$ $f(g(7)) = f(6) = 9$

Thus,

$f \circ g = \{(3, 9), (5, 12), (8, 7), (7, 9)\} = h.$ ✓

<u>Sample 2</u>. $g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$

$f =$ ____________________

$h = \{(3, 9), (5, 12), (8, 4), (7, 6)\}$

<u>Solution.</u> We are looking for a function f such that $f(6) = 9$, $f(9) = 12$, $f(4) = 7$, and $f(6) = 10$. That is, a function which contains the pairs $(6, 9)$, $(9, 12)$, $(4, 7)$, and $(6, 10)$. There is no such function. Why?

1. $g = \{(1, 1), (2, 1), (3, 1), (4, 2), (5, 3), (6, 3), (7, -1)\}$

   $f =$ ____________________________________

   $h = \{(1, 1), (2, 1), (3, 1), (4, 4), (5, 9), (6, 9), (7, 1)\}$

2. $g = \{(2, 5), (3, 8), (6, 8), (5, 0)\}$

   $f =$ ____________________

   $h = \{(2, 12), (3, 18), (6, 14), (5, 2)\}$

3. $g = \{(0, 1), (1, 5), (2, 9), (3, 8), (4, 9)\}$

   $f =$ ______________________

   $h = \{(1, 7), (2, 12), (4, 12)\}$

4. $g = \{(0, 1), (1, 5), (2, 9)\}$

   $f =$ ______________

   $h = \{(1, 5), (2, 8), (4, 8)\}$

5. $g = \{(3, 6), (8, 6)\}$

   $f =$ __________

   $h = \{(3, 9)\}$

6. $g = \{(x, y):\ y = x\}$

   $f =$ ______________

   $h = \{(x, y):\ y = x^2\}$

7. $g = \{(x, y):\ y = |x|\}$

   $f =$ ______________

   $h = \{(x, y):\ y = x\}$

8. $g = \{(x, y):\ y = x^2\}$

   $f =$ ______________

   $h = \{(x, y):\ y = x\}$

9. $g = \{(x, y):\ y = x^2\}$

   $f =$ ______________

   $h = \{(x, y):\ y = |x|\}$

10. $g = \{(x, y):\ xy = 1\}$

    $f =$ ______________

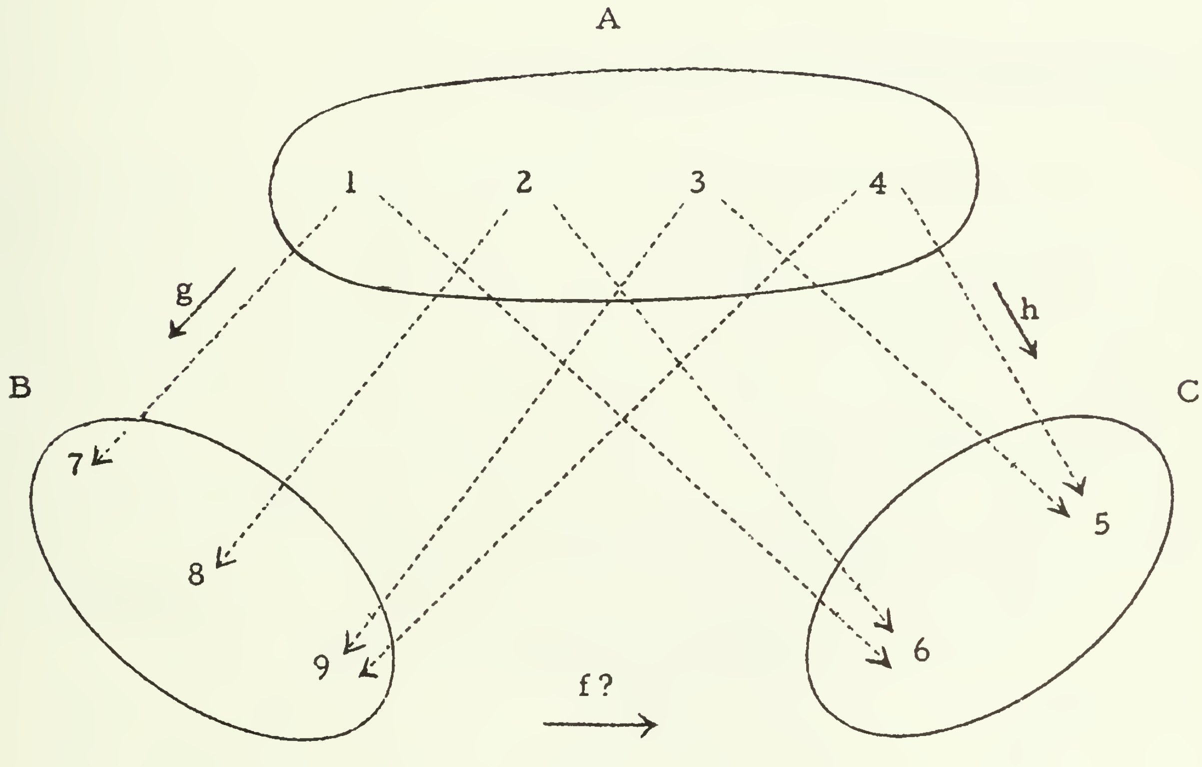
    $h = \{(x, y):\ y = x\}$

11. $g = \{(x, y):\ xy = 1\}$

    $f =$ ______________

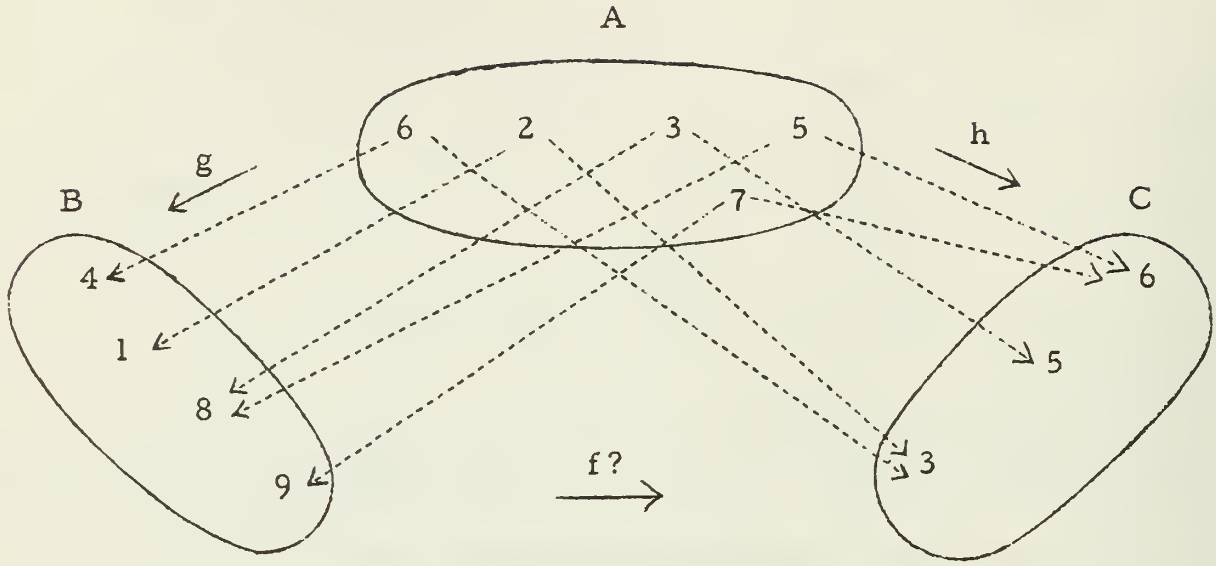
    $h = \{(x, y):\ x \neq 0 \text{ and } y = x\}$

12. Here is a picture which shows a mapping g of A on B, and a mapping h of A on C.



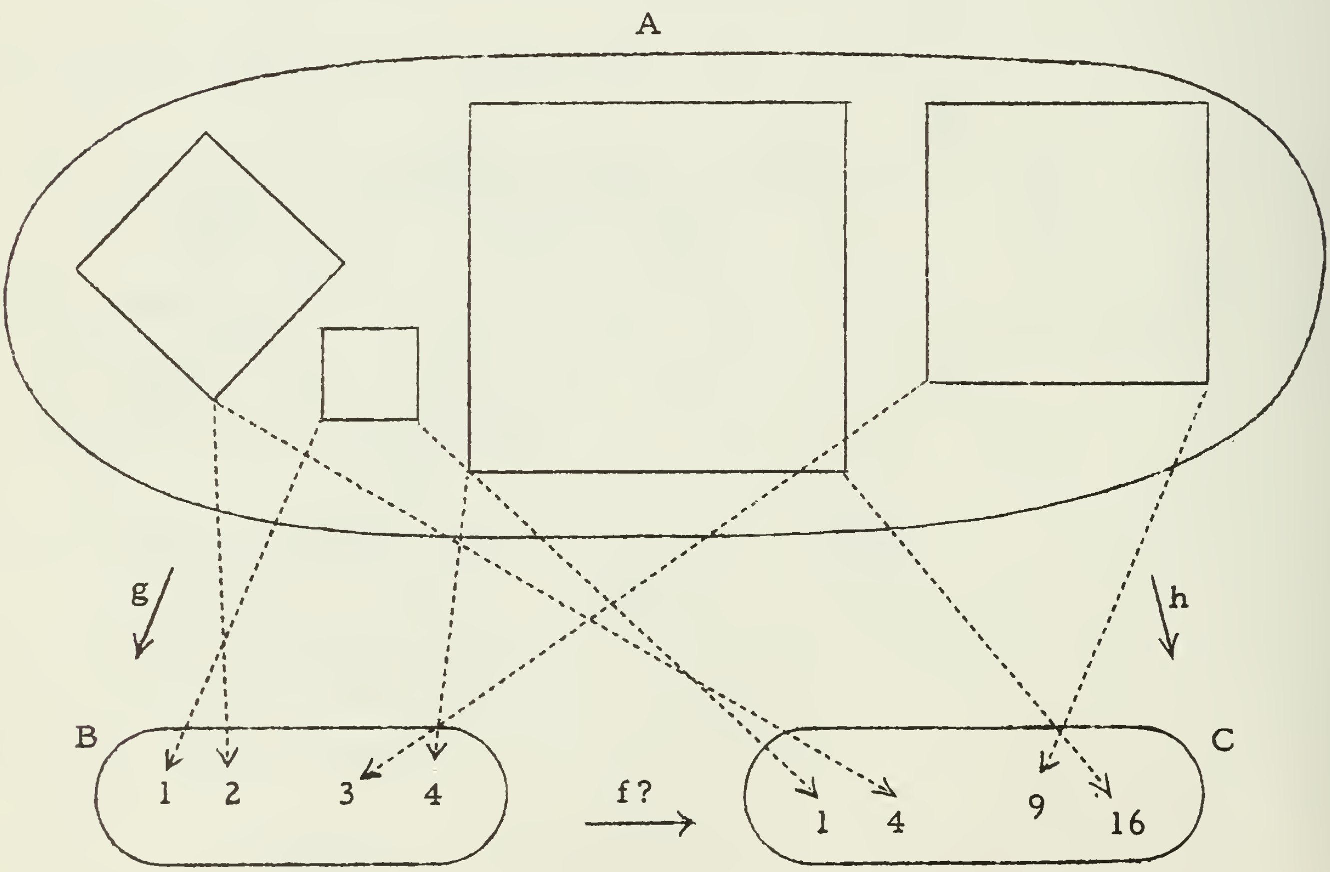
$f =$ ____

13. Here is a picture which shows a mapping g of A on B, and a mapping h of A on C.



f = ____

14. [The members of A are squares.]



f = ____

15. $g(x)$ = the side-measure of square x, $\mathcal{D}_g$ = the set of all squares

$f(x)$ = ____________ $\mathcal{D}_f$ = ____________

$h(x)$ = the zrea-measure of square x, $\mathcal{D}_h$ = the set of all squares

16. $g = \{(3, 5), (8, 2), (9, 5), (6, 0), (7, 5), (11, 11)\}$

f = ____________

$h = \{(3, 5), (8, 2), (9, 5), (6, 0), (7, 5), (11, 11)\}$

17. $g(x)$ = the area-measure of square x, $\mathcal{D}_g$ = the set of all squares

$f(x)$ = ____________ $\mathcal{D}_f$ = ____________

$h(x)$ = the side-measure of square x, $\mathcal{D}_h$ = the set of all squares

☆ 18. $g(x)$ = the husband or wife of x, $\mathcal{D}_g$ = the set of married people

$f(x)$ = ____________ $\mathcal{D}_f$ = ____________

$h(x)$ = the mother-in-law of x, $\mathcal{D}_h$ = the set of married people

FUNCTIONAL DEPENDENCE

Given a function g and a function f, you have seen that there is a function $f \circ g$, obtained by composing f with g. The domain of $f \circ g$ consists of those members x of $\mathcal{D}_g$ such that $g(x) \in \mathcal{D}_f$. As you have seen in the preceding Exploration Exercises, given functions g and h, it is sometimes possible to find a function f such that $h = f \circ g$. When there is such a function, we say that h <u>is functionally related to</u> g, or, for short, that

$$h \text{ is a function of } g.$$

In such a case, $\mathcal{D}_h \subseteq \mathcal{D}_g$ and the value of h for each of its arguments is determined by the value of g for that argument. So, one sometimes says that h <u>depends only on</u> g.

To say that h is a function of g is to claim that there is some function f such that $h = f \circ g$. One way to support such a claim is to discover such a function. [This you did several times in the Exploration Exercises.] Let's find another way.

To say that there is a function f such that $h = f \circ g$ is to say two things:

1. Each argument of h has a g-value [that is, $\mathcal{D}_h \subseteq \mathcal{D}_g$]
2. There is a set f of ordered pairs such that
   (a) f is a function which maps the g-value of each argument of h onto the h-value of this argument [that is, $f(g(x)) = h(x)$ for each $x \in \mathcal{D}_h$], and
   (b) h and $f \circ g$ have the same domain.

Notice, however, that if 2(a) is satisfied then each argument of h is sure to be an argument of $f \circ g$. So, in this case, 2(b) can be replaced by:

(b_1) each argument of $f \circ g$ is an argument of h.

Suppose that g and h satisfy condition 1, and that we have a table which lists, for each argument of h, the corresponding values of g and h.

arguments of h	values of g	values of h
⋮	⋮	⋮
a	g(a)	h(a)
⋮	⋮	⋮
b	g(b)	h(b)
·	·	·

Now, a set of ordered pairs which satisfies 2(a) is just a function which contains each of the ordered pairs of values of g and h listed in the table. If there is such a function then the set f_0 of listed ordered pairs will, itself, be such a function. And, if there is such a function which also satisfies 2(b_1), then f_0 will satisfy 2(b_1) as well as 2(a). On the other hand, since f_0 is the set of listed pairs, f_0 is bound to satisfy 2(a) if it is a function.

Summarizing the discussion so far, we see that h is a function of g if and only if

(1) $\mathcal{D}_h \subseteq \mathcal{D}_g$,

(2) f_0 [the set of listed ordered pairs] is a function,

and (3) each argument of $f_0 \circ g$ is an argument of h.

Now, by the definition of function, f_0 is a function if and only if there are not two arguments of h whose g-values are the same but whose h-values are different. [Exercise 2 on page 5-86 is an example of functions g and h which do not meet this condition.] So, condition (2) is satisfied if and only if

(a) for each two arguments x_1 and x_2 of h, if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.

Exercise 5 on page 5-86 shows that even if conditions (1) and (a) are satisfied, condition (3) may fail to be satisfied. In this exercise, $g = \{(3, 6), (8, 6)\}$ and $h = \{(3, 9)\}$. So, $\mathcal{D}_h = \{3\} \subseteq \{3, 8\} = \mathcal{D}_g$ and, since h has only one argument, (a) is certainly satisfied. However, in this case, $f_0 = \{(6, 9)\}$ and $f_0 \circ g = \{(3, 9), (8, 9)\}$. So, there is an argument [8] of $f_0 \circ g$ which is not an argument of h.

To see what it means for (3) to be satisfied, let's recall what the arguments of $f_0 \circ g$ are. The domain of f_0 is the set of listed values of g. So, by the definition of function composition, the domain of $f_0 \circ g$ is the set of arguments of g whose g-values are listed in the table. A value of g is listed in the table if and only if it is the g-value of some argument of h, that is, if and only if it is $g(x_1)$ for some $x_1 \in \mathcal{D}_h$. So,

x is an argument of $f_0 \circ g$

if and only if

$x \in \mathcal{D}_g$ and there is an $x_1 \in \mathcal{D}_h$ such that $g(x_1) = g(x)$.

Thus, to say that

if x is an argument of $f_0 \circ g$ then x is an argument of h

amounts to saying that

if $x \in \mathcal{D}_g$ and there is an $x_1 \in \mathcal{D}_h$ such that $g(x_1) = g(x)$ then $x \in \mathcal{D}_h$.

Hence, making use of the fact that $\mathcal{D}_h \subseteq \mathcal{D}_g$, we see that condition (3) is satisfied if and only if

(b) for each two arguments x_1 and x_2 of g, if $g(x_1) = g(x_2)$, and $x_1 \in \mathcal{D}_h$, then $x_2 \in \mathcal{D}_h$.

Finally, (b) and (a) can be combined into:

for all x_1 and x_2 in $\mathcal{D}_g$ such that $g(x_1) = g(x_2)$, if either x_1 or x_2 belongs to $\mathcal{D}_h$ then both belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$

So, we have established the following theorem:

For each function h, for each function g, there is a function f such that $h = f \circ g$

if and only if

$\mathcal{D}_h \subseteq \mathcal{D}_g$ and, for all x_1 and x_2 in $\mathcal{D}_g$ such that $g(x_1) = g(x_2)$, if either x_1 or x_2 belongs to $\mathcal{D}_h$ then both belong to $\mathcal{D}_h$ and $h(x_1) = h(x_2)$

Here is an example of the use of this theorem. Suppose

$$g = \{(0, 6), (1, 3), (3, 8), (4, 6), (5, 3), (9, 7)\}$$

and $h = \{(0, 7), (3, 2), (4, 7), (9, 8)\}$. Is h a function of g? Clearly, $\mathcal{D}_h \subseteq \mathcal{D}_g$. So, we look for pairs of arguments of g which have the same g-value. It turns out that the only such pairs are 0 and 4 $[g(0) = 6 = g(4)]$ and 1 and 5 $[g(1) = 3 = g(5)]$. Both 0 and 4 belong to $\mathcal{D}_h$ and $h(0) = 7 = h(4)$. Neither 1 nor 5 belong to $\mathcal{D}_h$. So, the conditions of the theorem are satisfied, and h is a function of g. [If we now wish to find a function f such that $h = f \circ g$ we know, without further checking, that f_o, that is, $\{(6, 7), (8, 2), (7, 8)\}$, is such a function. This is the function you would have tried had you been asked to do this example before proving the theorem. At that time you would have had to check to see that $f_o \circ g$ actually is h. In proving the theorem you have shown that, if the conditions given in the theorem are satisfied then $f_o \circ g$ is h, and there is no need for more checking. Also, if the conditions given in the theorem are not satisfied then there is no function f such that $h = f \circ g$.]

Here is a second example. Suppose

$$g = \{(0, 6), (1, 3), (3, 8), (4, 6), (5, 3), (9, 7)\}$$

and $h = \{(0, 7), (1, 10), (3, 2), (4, 7), (9, 8)\}$. Is h a function of g? Again, $\mathcal{D}_h \subseteq \mathcal{D}_g$, and the only pairs of arguments of g which have the same g-value are 0 and 4, and 1 and 5. Both 0 and 4 belong to $\mathcal{D}_h$, and $h(0) = h(4)$. But $1 \in \mathcal{D}_h$ and $5 \notin \mathcal{D}_h$. So, the conditions of the theorem are not satisfied, and h is not a function of g. [In this case, $f_o = \{(6, 7), (3, 10), (8, 2), (7, 8)\}$. Composing f_o with g, we see that

$$[f_o \circ g](0) = 7, \qquad [f_o \circ g](1) = 10, \qquad [f_o \circ g](3) = 2,$$
$$[f_o \circ g](4) = 7, \qquad [f_o \circ g](5) = 10, \qquad [f_o \circ g](9) = 8.$$

So, $f_o \circ g = \{(0, 7), (1, 10), (3, 2), (4, 7), (5, 10), (9, 8)\}$. As is to be expected from the proof, 5 is an argument of $f_o \circ g$. This is so because $g(5) = g(1)$ and $g(1)$ [that is, 3] is an argument of f_o. But, $5 \in \mathcal{D}_h$. Hence, $f_o \circ g \neq h$.]

EXERCISES

A. Use the test for functional dependence to answer the questions in each of the following exercises.

Sample 1. $g = \{(3, 8), (4, 0), (6, 5), (9, 8), (7, 0), (1, 0)\}$
$h = \{(4, 7), (6, 1), (7, 7), (1, 7)\}$

(a) Is g a function of h? (b) Is h a function of g?

Solution. (a) Since $\mathcal{D}_g \not\subseteq \mathcal{D}_h$, g is not a function of h.

(b) Since $\mathcal{D}_h \not\subseteq \mathcal{D}_g$, we investigate those arguments of g for which g has the same value. These are 3 and 9, and 4, 7, and 1, because

$$g(3) = g(9) = 8, \text{ and } g(4) = g(7) = g(1) = 0.$$

Now, 3 and 9 do not belong to $\mathcal{D}_h$. On the other hand, 4, 7, and 1 all belong to $\mathcal{D}_h$, and $h(4) = 7 = h(7) = h(1)$. So, according to the test for functional dependence, h is a function of g.

Sample 2. $p = \{(x, y): y = 2x + 5\}$, $q = \{(x, y), x \geq 0: y = x^2\}$
Is q a function of p?

Solution. Since $\mathcal{D}_q \subseteq \mathcal{D}_p$, q is a function of p unless there are two arguments of p for which p has the same value and such that either one of them belongs to $\mathcal{D}_q$ and the other does not, or both belong to $\mathcal{D}_q$ and the values of q for these arguments are different. But, p does not have the same value for any two of its arguments [that is, p has an inverse]. So, q is a function of p.
[Is p a function of q? Is $\{(x, y): x \geq 0 \text{ and } y = 2x + 5\}$ a function of q?]

1. $k = \{(0, 1), (1, 2), (2, 3), (4, 3), (5, 2), (6, 1)\}$
   $\ell = \{(6, 7), (5, 4), (4, 3), (2, 4), (1, 4), (0, 7)\}$
   (a) Is ℓ a function of k? (b) Is k a function of ℓ?
   (c) What two ordered pairs can you remove from ℓ so that the resulting function will be a function of k?
   (d) How can you obtain a function of k by changing the value of ℓ for just one of its arguments?

2. $u = \{(2, 4), (3, 2), (5, 4), (7, 4)\}$
   $v = \{(2, 7), (3, 9), (5, 7)\}$
   (a) Is v a function of u? (b) Is u a function of v?

3. $f = \{(x, y): y = 3x - 4\}$, $g = \{(x, y): y = -2x + 7\}$
   Is g a function of f?

4. $t = \{(x, y): y = 3\}$, $s = \{(x, y): y = 4x + 1\}$
   (a) Is s a function of t? (b) Is t a function of s?

5. $g = \{(x, y), y \geq 0: x^2 + y^2 = 25\}$, $h = \{(x, y), y \leq 0: x^2 + y^2 = 25\}$
Is h a function of g?

6. $c = \{(x, y) \in \text{Circles} \times N$: y is the circumference of $x\}$
$r = \{(x, y) \in \text{Circles} \times N$: y is the radius of $x\}$
(a) Is r a function of c? (b) Is c a function of r?

7. $P = \{(x, y) \in \text{Rectangles} \times N$: y is the perimeter of $x\}$
$A = \{(x, y) \in \text{Rectangles} \times N$: y is the area-measure of $x\}$
(a) Is A a function of P? (b) Is P a function of A?

8. $P = \{(x, y) \in \text{Squares} \times N$: y is the perimeter of $x\}$
$A = \{(x, y) \in \text{Squares} \times N$: y is the area-measure of $x\}$
(a) Is A a function of P? (b) Is P a function of A?

* * *

If g and h are functions for which there exists a function f such that $h = f \circ g$ then there are many such functions f. [Given one such function, you can get others by adding to it ordered pairs whose first components are not values of g.] However, there is a "smallest" such function, the set of all those ordered pairs $(g(x), h(x))$ such that $x \in \mathcal{D}_h$. It is the only function such that $h = f \circ g$ and $\mathcal{D}_f \subseteq \mathcal{R}_g$.

In Sample 2 of Part A on page 5-93 you have seen that if g is a function which has an inverse then, if h is any function such that $\mathcal{D}_h \subseteq \mathcal{D}_g$, h is a function of g. In this case it is particularly easy to find the function f such that $\mathcal{D}_f \subseteq \mathcal{R}_g$ and $h = f \circ g$. To see how, we solve the equation '$h = f \circ g$' for 'f'. To begin with, we compose each side with the inverse of g, and then use the fact that composition is associative.

$$h \circ g^{-1} = [f \circ g] \circ g^{-1} = f \circ [g \circ g^{-1}]$$

But, the domain of $g \circ g^{-1}$ is the domain of g^{-1}, and this is $\mathcal{R}_g$. And, for each $x \in \mathcal{R}_g$, $[g \circ g^{-1}](x) = x$. So, since $\mathcal{D}_f \subseteq \mathcal{R}_g$, the domain of $f \circ [g \circ g^{-1}]$ is $\mathcal{D}_f$ and, for each $x \in \mathcal{D}_f$, $[f \circ [g \circ g^{-1}]](x) = f([g \circ g^{-1}](x)) = f(x)$. Hence, $f \circ [g \circ g^{-1}] = f$. Consequently, [since $f \circ [g \circ g^{-1}] = h \circ g^{-1}$], $f = h \circ g^{-1}$. So, if g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$, you can find an f such that $h = f \circ g$ merely by composing h with the inverse of g.

Example 1. Consider the function g, where

$$g = \{(3, 6), (5, 9), (7, 5), (8, 4)\}$$

This function g has an inverse, and

$$g^{-1} = \{(6, 3), (9, 5), (5, 7), (4, 8)\}.$$

Now, consider any function h such that $\mathcal{D}_h \subseteq \mathcal{D}_g$, say,

$$h = \{(3, 9), (5, 12), (8, 7)\}.$$

Since

$$h \circ g^{-1} = \{(6, 9), (9, 12), (4, 7)\},$$

$\{(6, 9), (9, 12), (4, 7)\}$ is a function f such that $h = f \circ g$. [It is easy to check that this is the case. Do so now.]

Example 2. Consider the function g, where

$$g = \{(x, y): y = 3x - 4\}.$$

The converse of g is $\{(x, y): x = 3y + 4\}$, that is, $\{(x, y): y = \frac{x + 4}{3}\}$. Since this set of ordered pairs is a function, g has an inverse, and

$$g^{-1} = \{(x, y): y = \frac{x + 4}{3}\}.$$

Now, consider any function h such that $\mathcal{D}_h = \mathcal{D}_g$, say,

$$h = \{(x, y): y = -2x + 7\}.$$

Since $\mathcal{R}_{g^{-1}} = \mathcal{D}_h$, $\mathcal{D}_{h \circ g^{-1}} = \mathcal{D}_{g^{-1}}$, and we can find a description for $h \circ g^{-1}$ merely by substituting. [It was in order for this to work that we chose h so that $\mathcal{D}_h = \mathcal{D}_g$.] Doing so, we find that

$$\begin{aligned} h \circ g^{-1} &= \{(x, y): y = -2(\frac{x + 4}{3}) + 7\} \\ &= \{(x, y): y = \frac{-2x + 13}{3}\}. \end{aligned}$$

To check that this is a function f such that $f \circ g = h$ is to check that $[h \circ g^{-1}] \circ g = h$. Since $\mathcal{R}_{g^{-1}} = \mathcal{D}_g = \mathcal{D}_h$, $\mathcal{D}_{h \circ g^{-1}} = \mathcal{D}_{g^{-1}} = \mathcal{R}_g$. Hence, the domain of $[h \circ g^{-1}] \circ g$ is $\mathcal{D}_g$. So, we can find a description for $[h \circ g^{-1}] \circ g$ merely by substituting. Doing so, we find that

$$\begin{aligned} [h \circ g^{-1}] \circ g &= \{(x, y): y = \frac{-2(3x - 4) + 13}{3}\} \\ &= \{(x, y): y = -2x + 7\} = h. \quad \checkmark \end{aligned}$$

[If we modify Example 2 by choosing for h a function such that $\mathcal{D}_h \subseteq \mathcal{D}_g$ but $\mathcal{D}_h \neq \mathcal{D}_g$, say, the function h where

$$h = \{(x, y), x > 0: y = -2x + 7\},$$

then the desired function f is still $h \circ g^{-1}$. But, in finding a description for this function we must take account of $\mathcal{D}_h$. In

this case,

$$h \circ g^{-1} = \{(x, y),\ \frac{x + 4}{3} > 0:\ y = -2(\frac{x + 4}{3}) + 7\}$$
$$= \{(x, y),\ x > -4:\ y = \frac{-2x + 13}{3}\}.]$$

* * *

B. In each of the following exercises you are given functions g and h such that g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$. Your job is to find $h \circ g$, and to check that it is a function f such that $f \circ g = h$.

1.
$g = \{(4, 8), (3, 7), (2, 6), (1, 5)\}$
$g^{-1} =$ ____________________
$h = \{(3, 9), (2, 14), (1, 2)\}$
$f = h \circ g^{-1} =$ ____________________
$f \circ g =$ ____________________

2.
$g = \{(x, y):\ y = 2x + 9\}$
$g^{-1} =$ ______________
$h = \{(x, y):\ y = 4x - 11\}$
$f = h \circ g^{-1} = \{(x, y):$ __________$\}$
$f \circ g = \{(x, y):$ __________$\}$

3.
$g = \{(x, y):\ 3x + 2y = 6\}$
$g^{-1} =$ ______________
$h = \{(x, y),\ x > 4:\ y = 6x - 10\}$
$f = h \circ g^{-1} =$ ____________________
$f \circ g =$ ____________________

4.
$g = \{(x, y):\ y = x + 1\}$
$g^{-1} =$ ______________
$h = \{(x, y):\ y \geq 0 \text{ and } x^2 + y^2 = 1\}$
$f = h \circ g^{-1} =$ ____________________
$f \circ g =$ ____________________

☆5.
$g = \{(x, y):\ y \geq 0 \text{ and } y^2 = x\}$
$g^{-1} =$ __________________
$h = \{(x, y),\ x \geq 0:\ y = \sqrt{x}\}$
$f = h \circ g^{-1} =$ ________________
$f \circ g =$ ________________

[Supplementary exercises are in Part Q, page 5-265.]

5.06 Variable quantities. --While the range of a function may be any set whatever, many of the more useful functions are numerical-valued, that is, have ranges which consist of numbers. Examples of such functions are:

> $\{(x, y) \in \text{Circles} \times N\text{: } y \text{ is the radius of } x\}$
> the area-measure of a square
> the number of people in a family on December 31, 1959
> the double of a number

Turn, now, to page 5-53 and, in Parts D, E, and F, tell which of the functions listed there have numerical values.

Numerical-valued functions are sometimes called variable quantities. [The function which is called 'the area-measure of a square' is a variable quantity. (Area is a quantity which "varies" from square to square). The domain of this variable quantity is the set of all squares. Its range is the set of nonzero numbers of arithmetic.]

DEPENDENCE

Consider the variable quantities A and s where

$$A = \{(x, y) \in \text{Squares} \times N\text{: } y \text{ is the area-measure of } x\},$$

and

$$s = \{(x, y) \in \text{Squares} \times N\text{: } y \text{ is the side-measure of } x\}.$$

The domain of both these variable quantities is the set of all squares. Since two squares have the same area-measure if and only if they are congruent, and since they have the same side-measure if and only if they are congruent, each of the variable quantities A and s is a function of the other [Explain].

These results are not surprising, for you have known for a long time that to find the value of A for a given square, all you do is multiply the corresponding value of s by itself. Also, to find the value of s for a given square, you compute the square root of the corresponding value of A. Thus, one function f such that $A = f \circ s$ is the squaring function for numbers of arithmetic,

$$\{(u, v) \in N \times N\text{: } v = u^2\}.$$

Since, for each square x, $s(x) > 0$, another function which will work just as well is

$$\{(u, v) \in N \times N: \ u > 0 \text{ and } v = u^2\}.$$

[How does this second function differ from the first?] Similarly, two functions, g, such that $s = g \circ A$ are the square rooting function for numbers of arithmetic,

$$\{(u, v) \in N \times N: \ v = \sqrt{u}\},$$

and [since for each square x, $A(x) > 0$] the function

$$\{(u, v) \in N \times N: \ u > 0 \text{ and } v = \sqrt{u}\}.$$

[Do you see that if f is a function such that $A = f \circ s$ and f has an inverse then $s = f^{-1} \circ A$? Hint. Solve '$A = f \circ s$' for 's'.]

EXERCISES

A. Given the variable quantities c and d where

$$c = \{(x, y) \in \text{Circles} \times N: \ y \text{ is the circumference of } x\},$$

and

$$d = \{(x, y) \in \text{Circles} \times N: \ y \text{ is the diameter of } x\}.$$

1. Is c a function of d? If not, tell why. If c is a function of d, complete the following:

$$c = f \circ d \text{ where } f = \{(u, v) \in N \times N: \ \_\_\_\_\_\}$$

2. Is d a function of c? If not, tell why. If d is a function of c, complete the following:

$$d = g \circ c \text{ where } g = \{(u, v) \in N \times N: \ \_\_\_\_\_\}$$

B. Repeat Part A for the variable quantities A and ℓ where

$$A = \{(x, y) \in \text{Rectangles} \times N: \ y \text{ is the area-measure of } x\},$$

and

$$\ell = \{(x, y) \in \text{Rectangles} \times N: \ y \text{ is the length-measure of } x\}.$$

C. A taxicab company charges 35 cents for the first two thirds of a mile, or fraction thereof, and 10 cents for each succeeding third of a mile or fraction thereof. Consider the variable quantities c and m where

$$c = \{(x, y) \in \text{Trips} \times N\colon \text{ y cents is the charge for trip } x\},$$

and

$$m = \{(x, y) \in \text{Trips} \times N\colon \text{ x is a trip of y miles}\}.$$

1. Is c a function of m? If not, tell why. If c is a function of m, draw a graph of such a function.

2. Is m a function of c? If not, tell why. If m is a function of c, draw a graph of such a function.

OPERATIONS ON VARIABLE QUANTITIES

In earlier units you became acquainted with various operations on numbers. Some of the operations are oppositing, squaring, absolute valuing, and reciprocating. Others are addition and multiplication. Operations like those in the first list are called <u>singulary operations</u>; such operations are applied to single numbers. Those in the second list are called <u>binary operations</u>; they are applied to ordered pairs of numbers. Since whenever an operation can be applied to a number or to an ordered pair of numbers, the result is unique [otherwise it wouldn't be called 'an operation'!], an operation is simply a function. So, for example, we could use 'op' as a name for the oppositing function and use functional notation to write things like:

$$\text{op}({}^{+}2) = {}^{-}2, \qquad \text{op}({}^{-}5) = {}^{+}5, \qquad \text{op}(x + y) = \text{op}(x) + \text{op}(y),$$

$$[\text{op} \circ \text{op}]({}^{-}8), = {}^{-}8, \qquad \text{op} \circ [\text{op} \circ \text{op}] = \text{op}$$

However, we use instead an operator '$-$' and write things like:

$$-{}^{+}2 = {}^{-}2, \qquad -({}^{-}5) = {}^{+}5, \qquad -(x + y) = -x + -y$$

$$- -{}^{-}8 = {}^{-}8, \qquad \forall_x - - -x = -x$$

In a similar way, composing a function f with a function g can be

thought of as operating on g. For example, suppose

$$g = \{(\text{Al}, {}^{+}5), (\text{Ned}, {}^{+}4), (\text{Charles}, 0)\}$$

and f is the oppositing function, op. Then

$$\text{op} \circ g = \{(\text{Al}, {}^{-}5), (\text{Ned}, {}^{-}4), (\text{Charles}, 0)\}.$$

Instead of writing 'op ∘ g', we shall write '−g'. Note that when we do this, we are using the operator '−' in a new way. Previously, it named an operation on real numbers; now, we are using it to name an operation on functions. This is a case of ambiguity which is usually resolved by the context. For example, referring to the function g given above, we note that

$$[-g](\text{Al}) = -g(\text{Al}) = -{}^{+}5.$$

The first '−' stands for the operation on functions. The second and third '−'s stand for the operation on real numbers.

We can now define the oppositing operation on functions.

For each function g, −g is the function such that

$$[-g](x) = -g(x),$$

for each $x \in \mathcal{D}_g$ such that g(x) has an opposite.

EXERCISES

A. Fill the blanks.

1. If $g = \{(x, y): y = 3x + 12\}$ then

   (a) $[-g](2) =$ ____ (b) $[-g](-4) =$ ____ (c) $[-g](0) =$ ____

   (d) $[[-g] \circ g](4) =$ ____ (e) $[-[g \circ g]](4) =$ ____ (f) $-g =$ ____

2. If $t = \{(x, y), x \geq 0: y = \sqrt{x}\}$ then $-t =$ ____________.

3. If $u = \{(x, y): y = 5x - 7\}$ then $u(79) + [-u](79) =$ __________.

4. If $k = \{(x, y): y = 10x - 2\}$ then the solution of '$k(a) = [-k](a)$' is _____.

☆5. If $\mathcal{R}_f$ consists of real numbers then $f = -f$ if and only if $\mathcal{R}_f =$ ___.

☆6. $f = -f$ if and only if ________________________.

B. 1. Follow the pattern of the definition of oppositing for functions, and complete the following definition of the square-rooting operation:

For each function g, $\sqrt{g}$ is the function such that

$$\sqrt{g}\,(x) = \_\_\_\_\_\_\_\_,$$

for each ____________________.

Now, fill the blanks in the following exercises.

2. If $g = \{(x, y):\ y = x - 5\}$ then

(a) $\sqrt{g}(9) = \_\_\_\_$ (b) $\sqrt{g}\,(\_\_\_) = 3$ (c) $\sqrt{g}\,(5) = \_\_\_\_$

(d) $\sqrt{g}(4) = \_\_\_\_$ (e) $\sqrt{g}\,(\_\_\_) = -2$ (f) $\sqrt{g} = \_\_\_\_\_\_$

3. If $k = \{(x, y):\ y = x^2\}$ then $\sqrt{k} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

4. If A is the area-measure of a square and s is the side-measure of a square then $\sqrt{A} = \_\_\_\_\_$.

5. If $t = \{(x, y):\ y = 3x - 10\}$ then

(a) $[-t](3) = \_\_\_\_$ (b) $\sqrt{-t}(3) = \_\_\_\_$ (c) $\sqrt{-t}(-5) = \_\_\_\_$

(d) $[-\sqrt{t}](-5) = \_\_\_\_$ (e) $\sqrt{-t}(4) = \_\_\_\_$ (f) $[-\sqrt{t}](3) = \_\_\_\_$

(g) $-\sqrt{t} = \_\_\_\_\_\_\_\_\_\_$ (h) $\sqrt{-t} = \_\_\_\_\_\_\_$

C. 1. Write definitions of the operations on functions listed below.

(a) squaring (b) absolute valuing (c) reciprocating

Now, fill the blanks in the following exercises.

2. If $g = \{(x, y):\ y = 7x - 3\}$ then

(a) $g^2(2) = \_\_\_\_$ (b) $g^2(-1) = \_\_\_\_$ (c) $g^2(\_\_\_) = 16$

(d) $|g|(7) = \_\_\_\_$ (e) $|g|(-3) = \_\_\_\_$ (f) $|g|(\_\_\_) = 25$

(g) $\frac{1}{g}(1) = \_\_\_\_$ (h) $[1/g](0) = \_\_\_\_$ (i) $[1/g](\_\_\_) = 0$

(j) $[1/g](\frac{3}{7}) = \_\_\_\_$ (k) $1/|g^2| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$

D. Suppose $g = \{(x, y):\ y = x\}$. Graph the functions listed below.

1. g 2. g^2 3. $|g|$ 4. $-g$ 5. $1/g$

6. What are the domains and ranges of these five functions?

ADDING AND MULTIPLYING VARIABLE QUANTITIES

As an example of the occurrence of operations on functions, consider the formula '$A = s^2$' which you use in finding the area-measure of a given square when you know its side-measure.

$$A = \{(x, y) \in \text{Squares} \times N: \ y \text{ is the area-measure of } x\},$$

and

$$s = \{(x, y) \in \text{Squares} \times N: \ y \text{ is the side-measure of } x\}.$$

When you use the formula to find the area-measure of a given square x, you find the value, $s(x)$, of s for this square, and then multiply this number, $s(x)$, by itself. Since the variable quantity s^2 is

$$\{(x, y) \in \text{Squares} \times N: \ y = (s(x))^2\},$$

this amounts to finding the value, $[s^2](x)$, of s^2 for this given square x. Since the formula tells you that the variable quantity A <u>is</u> the variable quantity s^2, you know that by computing $[s^2](x)$, you have found the area-measure $A(x)$ of square x. Similarly, when given the area-measure of a square, you use the variable quantity $\sqrt{A}$ in order to find the side-measure of the square. The formula '$s = \sqrt{A}$' tells you that this is a correct procedure.

Formulas such as '$A = \ell w$' and '$P = 2(\ell + w)$', for the area-measure and perimeter of a rectangle, indicate that variable quantities can be multiplied and added. The definitions of these operations are obvious enough.

For all variable quantities g and h,

$$g + h = \{(x, y), \ x \in \mathcal{D}_g \cap \mathcal{D}_h: \ y = g(x) + h(x)\},$$

and

$$gh = \{(x, y), \ x \in \mathcal{D}_g \cap \mathcal{D}_h: \ y = g(x) \cdot h(x)\}.$$

In finding the area-measure of a rectangle x, you multiply $\ell(x)$ by $w(x)$. This is nothing more than computing the value, $[\ell w](x)$, of the variable quantity ℓw for its argument x.

<u>Example 1</u>. If $f = \{(3, 5), (7, 8), (5, 9), (8, 5), (2, 0)\}$ and
$g = \{(3, 9), (7, 2), (4, 3), (8, 16), (5, 7)\}$,
what are $f + g$ and fg?

<u>Solution</u>. According to the definitions, $f + g$ and fg are variable

quantities whose domain is the intersection of the domains of f and g. Applying the definition we see that

$$f + g = \{(3,\ 5 + 9),\ (7,\ 8 + 2),\ (5,\ 9 + 7),\ (8,\ 5 + 16)\}$$

$$= \{(3,\ 14),\ (7,\ 10),\ (5,\ 16),\ (8,\ 21)\},$$

and

$$fg = \{(3,\ 45),\ (7,\ 16),\ (5,\ 63),\ (8,\ 80)\}.$$

Example 2. If $f = \{(x, y),\ x > 0:\ y = 2x\}$ and
$g = \{(x, y),\ x < 5:\ y = x - 4\}$,
what are $f + g$ and fg?

Solution. The intersection of the domains of f and g is
$\{x:\ x > 0\} \cap \{x:\ x < 5\}$,

that is,

$$\{x:\ 0 < x < 5\}.$$

So,

$$f + g = \{(x, y),\ 0 < x < 5:\ y = (2x) + (x - 4)\}$$

$$= \{(x, y),\ 0 < x < 5:\ y = 3x - 4\},$$

and

$$fg = \{(x, y),\ 0 < x < 5:\ y = 2x^2 - 8x\}.$$

Example 3. Given f and g of Example 2, complete these sentences.

(a) $[f + g](3) =$ __________ (b) $[fg](3) =$ __________

(c) $[f + g](\frac{3}{4}) =$ __________ (d) $[fg](\sqrt{2}) =$ __________

(e) $[f + g](-\frac{1}{2}) =$ __________ (f) $[fg](6) =$ __________

Solution. (a) $[f + g](3) = 3 \cdot 3 - 4 = 5$ (b) $[fg](3) = 2 \cdot 3^2 - 8 \cdot 3 = -6$

(c) $[f + g](\frac{3}{4}) = 3 \cdot \frac{3}{4} - 4 = -\frac{7}{4}$

(d) $[fg](\sqrt{2}) = 2(\sqrt{2})^2 - 8\sqrt{2} = 4 - 8\sqrt{2}$

(e) and (f) '$[f + g](-\frac{1}{2})$' and '$[fg](6)$' are nonsense. [Why?]

* * *

Consider, again, the formula '$P = 2(\ell + w)$'. When using this formula, you obtain values of P by multiplying the number 2 by corresponding values of $\ell + w$. This is just what you would do if you were

finding values of the product of the variable quantity

$$\{(x, y),\ x \in \mathcal{D}_{\ell + w} :\ y = 2\}$$

by the variable quantity $\ell + w$. So, we could replace the formula

$$P = 2(\ell + w)$$

by the formula:

$$P = \{(x, y),\ x \in \mathcal{D}_{\ell + w} :\ y = 2\} \cdot (\ell + w)$$

This amounts to interpreting the symbol '2' as a name for the variable quantity $\{(x, y),\ x \in \mathcal{D}_{\ell + w} :\ y = 2\}$. If we do so, operations on real-valued variable quantities satisfy principles just like our basic principles for real numbers, and we can manipulate formulas according to rules just like those which we use in manipulating numerical equations. So, whenever we have a formula in which both numerals and names of variable quantities occur, we shall interpret each numeral as standing for a variable quantity whose domain is that of the other variable quantities and whose range is the set consisting of the number named by the numeral. [Variable quantities which are named by numerals are sometimes called <u>constant variable quantities</u>, or for short, <u>constants.</u>]

* * *

<u>Example 4.</u> If $f = \{(x, y):\ y = 3x + 7\}$, what are $9f$ and $f + 9$?

<u>Solution.</u> We interpret '9' as standing for a variable quantity whose range is $\{9\}$. Since $\mathcal{D}_f$ is the set of real numbers, we say that 9 is $\{(x, y):\ y = 9\}$. Then, by the definitions of multiplication and addition for variable quantities,

$$\begin{aligned} 9f &= \{(x, y):\ y = 9\} \cdot \{(x, y):\ y = 3x + 7\} \\ &= \{(x, y):\ y = 9(3x + 7)\} \\ &= \{(x, y):\ y = 27x + 63\}, \end{aligned}$$

and

$$\begin{aligned} f + 9 &= \{(x, y):\ y = 3x + 7\} + \{(x, y):\ y = 9\} \\ &= \{(x, y):\ y = (3x + 7) + 9\} \\ &= \{(x, y):\ y = 3x + 16\}. \end{aligned}$$

EXERCISES

A. Given $h = \{(3, 2), (7, 4), (0, 5), (5, -6), (8, 13)\}$
and $j = \{(5, 8), (8, 5), (7, -3), (2, 3), (6, 0), (9, 18)\}$.

1. What is $h + j$?
2. What is hj?
2. What is $j + h$?
4. What is jh?
5. What is $7 + h$?
6. What is $3j$?
7. What is h^2?
8. $[j + h](7) =$ __________
9. $[hj](5) =$ __________
10. $[j + 2h](8) =$ __________
11. $[jj](5) =$ __________
12. $[2j + h](8) =$ __________
13. $j(j(5)) =$ __________
14. Prove that $h + j \neq h \cup j$.

B. Given $c = \{(x, y): y = 2x + 7\}$ and $d = \{(u, v): v = 3u - 5\}$.

1. What is $c + d$?
2. What is cd?
3. $[c + d](2) =$ __________
4. $[cd](5) =$ __________
5. $[c + (-\frac{2}{3})d](9) =$ __________
6. $[c + (-\frac{2}{3})d](183) =$ __________
7. What is c^2?
8. What is d^2?

C. Suppose $f = \{(x, y): y = x\}$ and $g = \{(x, y): y = x + 3\}$.

1. Graph f and g on the same chart.
2. Graph $f + g$ on this chart. [Be lazy about it!]
3. Graph fg on this chart.

D. Suppose $f = \{(x, y): y = 7x - 9\}$.

1. $[f^2](2) =$ __________
2. $[f^2 + f](2) =$ __________
3. $[f^2 + 3](2) =$ __________
4. $[f^2 + 4f](2) =$ __________
5. $[4f + 3](2) =$ __________
6. $[f^2 + 4f + 3](2) =$ __________
7. $[f + 1](2) =$ __________
8. $[f + 3](2) =$ __________
9. $[(f + 1)(f + 3)](2) =$ __________
10. $[(f + 1) + (f + 3)](2) =$ __________

*

Here are definitions for subtracting and dividing variable quantities:

For all variable quantities g and h,

$$g - h = \{(x, y),\ x \in \mathcal{D}_g \cap \mathcal{D}_h:\ y = g(x) - h(x)\},$$

and $$\frac{g}{h} = \{(x, y),\ x \in \mathcal{D}_g \cap \mathcal{D}_h \text{ and } h(x) \neq 0:\ y = \frac{g(x)}{h(x)}\}.$$

*

E. Suppose $f = \{(1, 4), (2, 9), (3, 6), (4, 7)\}$

and $g = \{(1, 2), (2, 3), (3, 2), (4, 0)\}$.

1. What is $f - g$?
2. What is $g - f$?
3. What is $\frac{g}{f}$? $\frac{f}{g}$?
4. What are the domain and range of f? Of g? Of $\frac{g}{f}$? Of $\frac{f}{g}$?

F. Suppose $f = \{(x, y),\ x > 0:\ y = 6x\}$ and $g = \{(x, y),\ x > 0:\ y = 2x\}$.

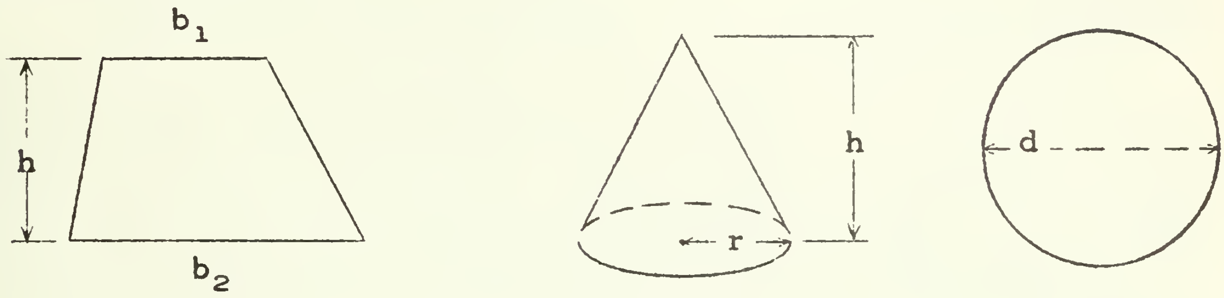
1. $[f - g](5)$ = ____________
2. $[f + g](5)$ = ____________
3. $[f - 3g](5)$ = ____________
4. $[f - 3g](78)$ = ____________
5. What is $f - 3g$?
6. What is $\frac{1}{3}f - g$?
7. $[\frac{f}{g}](5)$ = ____________
8. $[\frac{f}{g}](27)$ = ____________
9. What is $\frac{f}{g}$?
10. What is $\frac{g}{f}$?

G. Consider the constants 2 and 3.

1. If x_1 and x_2 are two arguments in a domain of 2, what are the values corresponding to these arguments? That is, what are $2(x_1)$ and $2(x_2)$?
2. Repeat Exercise 1 for the constant 3.
3. Suppose 2 and 3 are constants with a common domain. Does it follow that $2 \cdot 3$ is a constant? How about $2 + 3$? $2 - 3$? $2 \div 3$?
4. If x_1 and x_2 are two arguments in a common domain of 2 and 3, what are the values of $2 + 3$ corresponding to these arguments? Does $[2 \cdot 3](x_1) = [2 \cdot 3](x_2)$?

FORMULAS

A formula is a statement about variable quantities. Read the formulas in the diagrams.



$A = \frac{1}{2}h(b_1 + b_2)$ $V = \frac{1}{3}\pi r^2 h$ $c = \pi d$

The first formula tells you in very concise form that the area-measure of a trapezoid is one half the product of its height-measure and the sum of its base-measures. The variable quantity A <u>is</u> the variable quantity $\frac{1}{2}h(b_1 + b_2)$. What are the domains of A, $\frac{1}{2}$, h, b_1, and b_2?

The second formula tells you that the variable quantity which is the volume-measure of a circular cone is the variable quantity $\frac{1}{3}\pi r^2 h$. What are the domains of the variable quantities V, r, and h? What is the domain of the variable quantity r^2? What are the domains of the constants $\frac{1}{3}$, π, and $\frac{1}{3}\pi$?

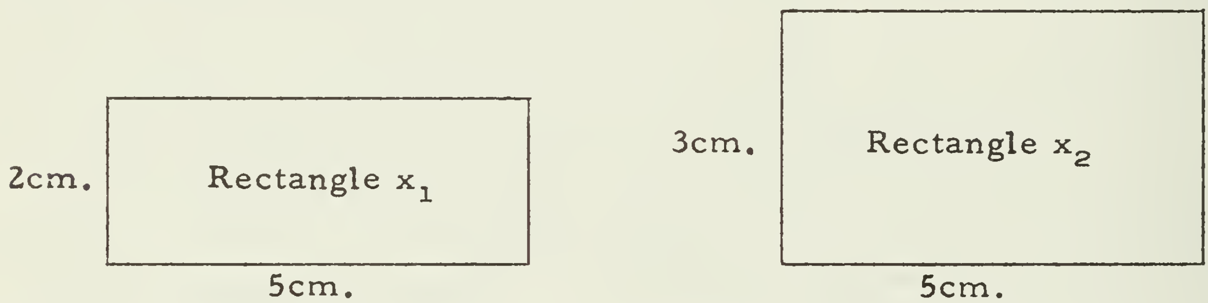
The third formula tells you that c is πd, that is, the variable quantity which is the circumference of a circle is the same variable quantity as the product of the constant π by the variable quantity which is the diameter of the circle. What is the quotient of the variable quantity c by the variable quantity d?

The formula '$c = \pi d$' tells you that, for each circle x, $c(x) = [\pi d](x)$. By the rule for multiplying variable quantities, we know that, for each circle x, $[\pi d]x = \pi(x) \cdot d(x)$. Since π is a constant, we know that, for each x, $\pi(x) = \pi$. So, for each circle x, $c(x) = \pi \cdot d(x)$. Consider two arguments in the domain of d, say, circle x_1 and circle x_2, such that $d(x_1) = d(x_2)$. [What do you know about the circles x_1 and x_2?] Does it follow that $c(x_1) = c(x_2)$? In that case, we have met the conditions [page 5-91]. which tell us that the variable quantity c is a function of the variable quantity d. So, there is a function f such that $c = f \circ d$. One such function f is $\{(u, v) \in N \times N: v = \pi u\}$.

We can use the formula '$c = \pi d$' both to discover that c is a function of d and to find a function f such that $c = f \circ d$. Since c is a function of d, we say that c depends only on d. We also say that c is the dependent variable quantity when d is the independent variable quantity.

From the formula for the area-measure of a square we learn that A is a function of s, so that A is the dependent variable quantity when s is the independent variable quantity. From the formula for the side-measure of a square ['$s = \sqrt{A}$'] we see that s depends only on A. So, s is the dependent variable quantity when A is the independent variable quantity.

Now, consider the formula which tells you that P is $2(\ell + w)$. We know that, for each rectangle x, $P(x) = 2(\ell(x) + w(x))$. Take two arguments in the common domain of P, ℓ, and w, say, rectangles x_1 and x_2.



Here $\ell(x_1) = \ell(x_2)$. Does it follow that $P(x_1) = P(x_2)$? Hence, does it follow that P is a function of ℓ? Do you think that P is a function of w? [Justify your answer.]

Suppose you have two rectangles, x_3 and x_4, such that $\ell(x_3) = \ell(x_4)$ and $w(x_3) = w(x_4)$. [What can you say about these rectangles?] Does it follow that $P(x_3) = P(x_4)$? In this case [in analogy with the case of one variable quantity being a function of another], we say that P is a function of (ℓ, w). This means that there is a function f such that, for each rectangle, x, $P(x) = f((\ell(x), w(x)))$. The range of such a function includes the range of P, and the domain of such a function contains the ordered pairs whose first and second components are the values of ℓ and w, respectively, for arguments of P. One such function f is

$$\{((u, v), w) \in (N \times N) \times N: \; w = 2(u + v)\}.$$

What is $f((5, 2))$? $f((5, 3))$? $f((9, 6)) = ?$ $f((3, ?)) = 10$ $f((?, 7)) = 34$

In the case of a variable quantity being a function of another, we say that the first is the dependent variable quantity and the second is the independent variable quantity. When a variable quantity is a function of an ordered pair of variable quantities, we say that the first is the dependent variable quantity and that the components of the pair are the independent variable quantities.

So, the formula '$P = 2(\ell + w)$' tells us that P depends only on ℓ and w, that is, that there is a function which can be applied to pairs of corresponding values of ℓ and w to produce the corresponding value of P. Functions like these whose domains consist of ordered <u>pairs</u> are sometimes called <u>functions of two variables</u> to contrast them with the functions we have been dealing with up to now which are called <u>functions of one variable</u>.

Consider the formula '$V = \frac{1}{3}\pi r^2 h$' for the volume of a circular cone. Is V a function of r? The answer is 'no' because there are cones, x_1 and x_2, such that $r(x_1) = r(x_2)$ and $V(x_1) \neq V(x_2)$. Can you describe two such cones? Is V a function of h? [Tell why.] Is V a function of π? Is V a function of $\frac{1}{3}$? Is V a function of (r, h)? Let's see. Suppose x_1 and x_2 are cones such that $(r(x_1), h(x_1)) = (r(x_2), h(x_2))$. Now,

$$V(x_1) = [\tfrac{1}{3}\pi](x_1) \cdot [r^2](x_1) \cdot h(x_1),$$

and

$$V(x_2) = [\tfrac{1}{3}\pi](x_2) \cdot [r^2](x_2) \cdot h(x_2).$$

Since $\frac{1}{3}\pi$ is a constant, $[\frac{1}{3}\pi](x_1) = [\frac{1}{3}\pi](x_2)$. Since r^2 is a function of r, and $r(x_1) = r(x_2)$, it follows that $[r^2](x_1) = [r^2](x_2)$. So, since $h(x_1) = h(x_2)$, it follows that $V(x_1) = V(x_2)$. Hence, V is a function of (r, h), that is, there is a function f such that, for each cone x, $V(x) = f(r(x), h(x))$. Such a function f is a function of two variables. One possible function is

$$\{((u, v), w) \in (N \times N) \times N: \ w = \tfrac{1}{3}\pi u^2 v\}.$$

Is V a function of (h, r)? What function? If there is such a function is it different from the function f mentioned above?

EXERCISES

A. Consider the variable quantity P where

$$P = \{(\text{Al}, 9), (\text{Bob}, 6), (\text{Cora}, 8)\}.$$

1. If the variable quantity K is $2P + 1$, what is the domain of K?

2. (a) $K(\text{Al}) =$ ______ (b) $K(\text{Bob}) =$ ______ (c) $K(\text{Cora}) =$ ______

3. (a) $[K^2](\text{Al}) =$ ____ (b) $[K^2 + 3K + 2](\text{Bob}) =$ ______

B. Consider the variable quantity M where

$$M = \{(1, 8), (2, 7), (3, 4), (4, 3)\}.$$

1. If the variable quantity N is the variable quantity $3M + 4$,

   (a) $N(2) =$ __________ (b) $N(4) =$ __________

   (c) $[N^2](3) =$ __________ (d) $[N^2 + 2N](1) =$ __________

2. If $K = 7M - 1$, list the ordered pairs in K.

3. If $J = 5M - 2$, is there a function f such that $J = f \circ M$? Find such an f.

4. (a) Suppose $A = f \circ M$ where $f = \{(x, y) : y = 3x^2 + 2x + 1\}$. List the ordered pairs which belong to A.

   (b) Is there a formula for A in terms of M? If there is, write one.

5. (a) Suppose $T = \{(1, 64), (2, 49), (3, 16), (4, 9)\}$. Is T a function of M? [That is, is there an f such that $T = f \circ M$?] What function? [That is, describe one such f.]

   (b) Can you write a formula for T in terms of M? Try to.

6. (a) Suppose $S = \{(1, 15), (2, 13), (3, 7), (4, 5)\}$. Is S a function of M? What function?

   (b) Can you write a formula for S in terms of M? Try to.

7. (a) Suppose $Q = \{(1, 98), (2, 347), (3, 629), (4, -708)\}$. Is Q a function of M? What function?

   ☆ (b) Can you write a formula for Q in terms of M? Try to, but don't try too hard.

8. (a) If $R = 3M + 5$, what function is R of M?

   (b) Is M a function of R? What function?

   (c) Can you write a formula for M in terms of R? Try to.

9. (a) Suppose that E is a variable quantity whose domain is that of M, and that each value of E is 5 less than the corresponding value of M. Is E a function of M? What function? Write (if you can) a formula for E in terms of M.

   (b) Is M a function of E? Write (if you can) a formula for M in terms of E.

   (c) List the ordered pairs in the variable quantity $\frac{E}{M}$.

   (d) Is $\frac{E}{M}$ a function of M? What function?

   (e) Write (if you can) a formula for $\frac{E}{M}$ in terms of M.

10. (a) Suppose $B = f \circ M$ where $f = \{(x, y): y = x\}$. Write a formula for B in terms of M.

    (b) Suppose $C = g \circ M$ where $g = \{(x, y): y = 3\}$. Write a formula for C in terms of M.

☆11. (a) Is the constant 2 a function of M?

    (b) Is M a function of the constant 2?

C. Suppose A and B are variable quantities where

$$A = \{(1, 5), (2, 6), (3, 6), (4, -5)\}.$$

and

$$B = \{(1, 11), (2, 13), (3, -11), (4, 13)\}.$$

1. If $C = 2A + 3B$ then

   (a) $C(1) =$ __________ (b) $C(2) =$ __________

   (c) $C(3) =$ __________ (d) $C(4) =$ __________

   (e) $[C^2](1) =$ __________ (f) $[C^2 - 5](4) =$ __________

2. (a) If $D = 2A - B$, what are the ordered pairs in D?

   (b) If $D = (A - B)(A + B)$, what are the ordered pairs in D?

   (c) If $D = A^2 - B^2$, what are the ordered pairs in D?

D. Suppose A and B are variable quantities such that $A = 3B$.

1. If x_1 and x_2 belong to the common domain of A and B, and if $B(x_2)$ is $B(x_1) + 5$, what can you say about $A(x_2)$ and $A(x_1)$?

2. If $B(x_2)$ is $7 \circ B(x_1)$, what can you say about $A(x_2)$ and $A(x_1)$?

[Supplementary exercises are in Part R, page 5-266.]

TRANSFORMING FORMULAS

Earlier we said that operations on real-valued variable quantities satisfy principles just like our basic principles for real numbers, and that this meant that you could manipulate formulas just like you manipulate numerical equations. Let's check up on these assertions. To do so, choose some set D and consider all real-valued variable quantities whose domain is D. Also, use '0' and '1' as names for the variable quantities whose domain is D and whose value for each argument is the real number 0, or the real number 1, respectively.

One of our basic principles for real numbers is the dpma:

$$\forall_x \forall_y \forall_z \, (x + y)z = xz + yz$$

So, one thing we want to check is the distributive principle for multiplication of variable quantities with respect to addition of variable quantities. This is:

$$\forall_f \forall_g \forall_h \, (f + g)h = fh + gh$$

We want to show that if f, g, and h are real-valued variable quantities with domain D, then $(f + g)h$ is the same variable quantitiy as $fh + gh$. This is easy. By the definitions of addition and multiplication of variable quantities, the domain of $f + g$ is $\mathfrak{D}_f \cap \mathfrak{D}_g$, and that of $(f + g)h$ is $\mathfrak{D}_{f+g} \cap \mathfrak{D}_h$. Since $\mathfrak{D}_f = \mathfrak{D}_g = \mathfrak{D}_h = D$, the domain of $(f + g)h$ is also D. Similarly, the domain of $fh + gh$ is D. But, if $x \in D$ then

$$\begin{aligned}
&[(f + g)h](x) && \\
&= [f + g](x) \cdot h(x) && \text{definition of multiplication} \\
&= \big(f(x) + g(x)\big) \cdot h(x) && \text{definition of addition} \\
&= f(x) \cdot h(x) + g(x) \cdot h(x), && \text{dpma for real numbers}
\end{aligned}$$

and

$$\begin{aligned}
&[fh + gh](x) && \\
&= [fh](x) + [gh](x) && \text{definition of addition} \\
&= f(x) \cdot h(x) + g(x) \cdot h(x), && \text{definition of multiplication}
\end{aligned}$$

also. So, since $(f + g)h$ and $fh + gh$ have the same domain and have the same value for each argument, $(f + g)h = fh + gh$.

In the same way, you can show that the commutative and associative principles for addition and multiplication of real numbers imply the

corresponding principles for addition and multiplication of real-valued variable quantities with domain D. So, we have:

$$\forall_f \forall_g\ f + g = g + f \qquad\qquad \forall_f \forall_g\ fg = gf$$

$$\forall_f \forall_g \forall_h\ f + g + h = f + (g + h) \qquad\qquad \forall_f \forall_g \forall_h\ fgh = f(gh)$$

For the same reason [that is, because operations on variable quantities amount to operations on their values], we also have principles for the variable quantities 0 and 1:

$$\forall_f\ f + 0 = f$$

$$\forall_f\ f \cdot 1 = f$$

and a principle of opposites and a principle for subtraction:

$$\forall_f\ f + -f = 0 \qquad\qquad \forall_f \forall_g\ f - g = f + -g$$

Finally, if, for each $x \in D$, $g(x) \neq 0$ then the domain of $f \div g$ is also D, and, for each $x \in D$,

$$\begin{aligned}
&[(f \div g)g](x) \\
&= [f \div g](x) \cdot g(x) && \text{[Why?]} \\
&= \big(f(x) \div g(x)\big) \cdot g(x) && \text{[Why?]} \\
&= f(x). && \text{[Why?]}
\end{aligned}$$

So, the principle of quotients for real-valued variable quantities with domain D is:

For each f, for each g such that 0 is not a value of g,

$$(f \div g)g = f.$$

This corresponds with the last of our basic principles for real numbers.

Our checking of the basic principles now pays us a big dividend. Each of the theorems about real numbers which you derived in Unit 2 from the basic principles for real numbers [there were 78 of them!] can be translated into a statement about real-valued variable quantities with domain D. <u>And</u>, you know that each of these statements is true, because it can be derived from the principles for variable quantities in just the same way that the corresponding theorem about real numbers is derived from the ten basic principles for real numbers.

For example, the cancellation principle for addition [for real numbers] translates into:

$$\forall_f \forall_g \forall_h \text{ if } f + h = g + h \text{ then } f = g$$

And, the test-pattern for the cancellation principle translates into:

Suppose that $f + h = g + h$

Then $f + h + -h = g + h + -h,$ [uniqueness principle]

$f + (h + -h) = g + (h + -h),$ [apa for variable quantities]

$f + 0 = g + 0$ [po for variable quantities]

and $f = g.$ [pa0 for variable quantities]

Hence, if $f + h = g + h$ then $f = g$.

As another example of translation, the division theorem [for real numbers] translates into:

(*) For each f, for each g such that 0 is not a value of g, for each h,

$$\text{if } hg = f \text{ then } h = f \div g.$$

On translating a proof of the division theorem, one obtains a proof of the displayed theorem concerning variable quantities.

Although we have been discussing real-valued variable quantities, all those theorems of Unit 2 which do not involve subtraction or opposition can be translated into theorems about variable quantities, with a given domain D, whose values are numbers of arithmetic. This is because all the basic principles for real numbers except the po and the ps hold for numbers of arithmetic and, so, the translations of them given above hold for variable quantities whose values are numbers of arithmetic. Thus, for example, (*) shows that the formula '$\pi = c \div d$' can be derived from the formula '$\pi d = c$'. [Notice that the first of these formulas would be nonsense if 'π' did not name a variable quantity.] Using the principle of quotients for variable quantities [together with the fact that no circle has diameter zero], one can derive the second formula from the first. So, the two formulas are equivalent.

On the basis of the results arrived at, one can use analogues of the equation transformation principles to solve a formula for one of

the letters occurring in it. For example, we can solve '$P = 2(\ell + w)$' for 'w'.

$$\begin{aligned} P &= 2(\ell + w) \\ P &= 2\ell + 2w \\ P - 2\ell &= 2w \\ w &= \frac{P - 2\ell}{2} \end{aligned}$$

addition transformation principle (from $P = 2\ell + 2w$ to $P - 2\ell = 2w$)

multiplication transformation principle (from $P - 2\ell = 2w$ to $w = \frac{P-2\ell}{2}$)

EXERCISES

Solve each of the following formulas for the indicated letter.

1. $P = a + b + c$; b
2. $P = 2a + b$; a
3. $V = \frac{1}{3}Bh$; h
4. $s = \frac{1}{2}(a + b + c)$; c
5. $V = \frac{1}{3}\pi r^2 h$; h
6. $m = \frac{1}{2}(b_1 + b_2)$; b_1
7. $A = \frac{1}{2}h(b_1 + b_2)$; b_1
8. $S = \frac{n}{2}(a + \ell)$; a
9. $S = \frac{n}{2}(a + \ell)$; n
10. $T = 2\pi rh + 2\pi r^2$; h
11. $C = \frac{E}{R + r}$; r
12. $S = \frac{a}{1 - r}$; r
13. $S = \frac{ar^n - a}{r - 1}$; a
14. $K = \frac{PV}{T}$; V
15. $S = \frac{2\pi dr^2}{d + r}$; d
16. $V = \frac{h}{6}(B_1 + B_2 + 4M)$; M

EXPLORATION EXERCISES

A. 1. Graph on the same chart the functions defined by these equations.

(a) $y = 3x$ (b) $y = \frac{1}{2}x$ (c) $y = -2x$

2. Give the set of ordered pairs which is the intersection of the three functions of Exercise 1.

3. Give the set of ordered pairs which is the intersection of the functions $\{(x, y)\colon\ y = ax\}$, for all a.

B. 1. Graph on the same chart the functions defined by these equations.

(a) $y = -2$ (b) $y = 7$ (c) $y = 0$

2. Give the set of ordered pairs which is the intersection of the three functions of Exercise 1.

3. Give the set of ordered pairs which is the intersection of any two of the functions $\{(x, y)\colon\ y = b\}$, for all b.

C. 1. Graph the function $g_1 + g_2$ where $g_1 = \{(x, y): y = 3x\}$ and $g_2 = \{(x, y): y = 2\}$.

2. Graph $\{(x, y): y = 3x + 2\}$.

D. 1. Graph on the same chart the functions defined by these equations.

(a) $y = 4x + 2$ (b) $y = -3x + 2$ (c) $y = \frac{2}{3}x + 2$

2. Repeat Exercise 1 for these equations.

(a) $y = 4x + 0$ (b) $y = -3x + 0$ (c) $y = -\frac{1}{2}x + 0$

3. Repeat Exercise 1 for these equations.

(a) $y = 4x - 5$ (b) $y = -6x - 5$ (c) $y = \frac{3}{5}x - 5$

E. In each of the following exercises you are given a pair of equations each of which defines a function. For each exercise, give the set of ordered pairs which is the intersection of the functions.

1. $y = 2x + 5$
   $y = 3x + 5$

2. $y = -3x - 7$
   $y = 6x - 7$

3. $y = 571x + 9$
   $y = -35x + 9$

4. $y = 84x$
   $y = -3x$

5. $y = 7x + 2$
   $y = 7x + 8$

6. $y = -5x + 3$
   $y = -5x - 7$

F. Give numerals for the frames to make true sentences.

1. $\{(x, y): y = 3x + \bigcirc\}$ contains the point (0, 7)

2. $(0, 83) \in \{(x, y): y = 2x + \bigcirc\}$

3. The graph of $\{(x, y): y = -5x + \bigcirc\}$ crosses the graph of the y-axis 12 units above the graph of the origin.

4. The graph of $\{(x, y): y = -\frac{1}{2}x + \bigcirc\}$ contains the graph of the origin.

5. $\{(x, y): y = \square x + 0\}$ contains the point (1, 5)

6. $\{(x, y): y = \square x + 0\}$ contains the point (1, 52)

7. $\{(x, y): y = \square x + 0\}$ contains the point (5, 1)

8. $(3, 6) \in \{(x, y): y = \square x\}$

9. $(6, 3) \in \{(x, y): y = \square x\}$

10. $(2, 5) \in \{(x, y): y = \square x\}$

11. $(2, -1) \in \{(x, y): y = \square x\}$

12. $(-1, -2) \in \{(x, y): y = \square x\}$ 13. $(-3, -9) \in \{(x, y): y = \square x\}$

14. $\{(x, y): y = 3x + \bigcirc\}$ contains the points (0, 2) and (1, $\triangle$)

15. $\{(x, y): y = \square x + \bigcirc\}$ contains the points (0, 2) and (1, 5)

16. $\{(x, y): y = \square x + \bigcirc\}$ contains the points (0, 2) and (1, 9)

17. $\{(0, 2), (1, 79)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

18. $\{(0, 2), (3, \triangle)\} \subseteq \{(x, y): y = 5x + \bigcirc\}$

19. $\{(0, 2), (3, 23)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

20. $\{(0, 2), (5, 22)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

21. $\{(0, 2), (5, -3)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

22. $\{(0, 8), (1, 10)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

23. $\{(0, -4), (2, 2)\} \subseteq \{(x, y): y = \square x + \bigcirc\}$

24. $\{(0, \triangle), (1, \bigtriangledown)\} \subseteq \{(x, y): y = 7x + 5\}$

25. $\{(0, \triangle), (2, \bigtriangledown)\} \subseteq \{(x, y): y = 9x + 3\}$

26. $\{(0, 329), (5, \square + 329)\} \subseteq \{(x, y): y = 42x + \bigcirc\}$

27. $\{(0, 4), (1, 4 + \square), (2, 4 + \bigcirc)\} \subseteq \{(x, y): y = 13x + \bigcirc\}$

28. $\{(0, 4), (1, 4 + \square), (2, 4 + \bigcirc)\} \subseteq \{(x, y): y = -13x + \bigcirc\}$

<u>G</u>. Complete to make true generalizations.

1. $\forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = 19x\}$

2. $\forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = -29x\}$

3. $\forall_a \forall_k \{(0, 0), (k, \_\_\_)\} \subseteq \{(x, y): y = ax\}$

4. $\forall_{k \neq 0} \forall_m \{(0, 0), (k, m)\} \subseteq \{(x, y): y = \_\_\_ \cdot x\}$

5. $\forall_a \forall_b (0, \_\_\_) \in \{(x, y): y = ax + b\}$

6. $\forall_b \{(0, \_\_\_), (1, \_\_\_\_\_)\} \subseteq \{(x, y): y = 7x + b\}$

7. $\forall_b \{(0, \_\_\_), (3, \_\_\_\_\_)\} \subseteq \{(x, y): y = 7x + b\}$

8. $\forall_a \forall_b \{(0, \_\_\_), (1, \_\_\_\_\_), (2, \_\_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

9. $\forall_a \forall_b \forall_p \{(0, \_\_\_), (p, b + \_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

10. $\forall_b \forall_p \{(1, \_\_\_\_), (1 + p, 9 + b + \_\_\_\_)\} \subseteq \{(x, y): y = 9x + b\}$

11. $\forall_b \forall_p \{(-7, \_\_\_\_), (-7+p, 14+b+\_\_\_\_)\} \subseteq \{(x, y): y = -2x+b\}$

12. $\forall_a \forall_b \forall_p \{(3, \_\_\_\_), (3+p, a3+b+\_\_\_\_)\} \subseteq \{(x, y): y = ax + b\}$

13. $\forall_a \forall_b \forall_p \forall_q \{(q, \_\_\_\_), (q+p, aq+b+\_\_\_\_)\} \subseteq \{(x, y): y = ax+b\}$

14. $\forall_a \forall_b \forall_s \forall_t$ if $(s, t) \in \{(x, y): y = ax + b\}$

then $\forall_p\ (s + p, t + \_\_\_) \in \{(x, y): y = ax + b\}$

5.07 Linear functions.--The domain of functions such as

$$\{(x, y): y = 2\}, \quad \{(x, y): y = -3\}, \quad \text{and} \quad \{(x, y): y = 43.5\},$$

is the set of real numbers. The range of each such function consists of a single real number. These functions are called constant functions of a real variable. We say 'of a real variable' because the domain is the set of real numbers. There are also constant functions of two [or more] real variables such as $\{((x, y), z): z = 3\}$. But, since we shall not discuss them in this section, it will not cause confusion if we abbreviate 'constant function of a real variable' to 'constant function'.

The graphs of constant functions are horizontal lines, that is, lines perpendicular to the graph of the y-axis.

Now, consider functions of a real variable such as

$$\{(x, y): y = 3x - 2\}, \quad \{(x, y): y = -2x+0\}, \text{ and } \{(x, y): y = \tfrac{1}{2}x + 5\}.$$

What are the domain and range of these functions? Notice that each of these functions is the sum of two functions, one of which is a constant function. For example,

$$\{(x, y): y = 3x - 2\} = \{(x, y): y = 3x\} + \{(x, y): y = -2\}.$$

Figure 1 shows the graph of $\{(x, y): y = 3x\}$; Figure 2 shows the graph

of $\{(x, y): y = -2\}$; and Figure 3 shows the graph of their sum.

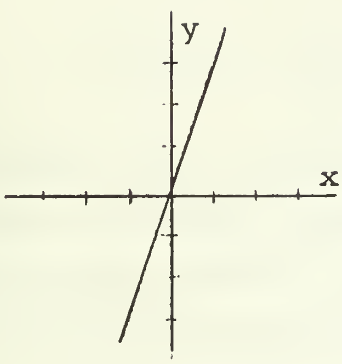


Fig. 1.

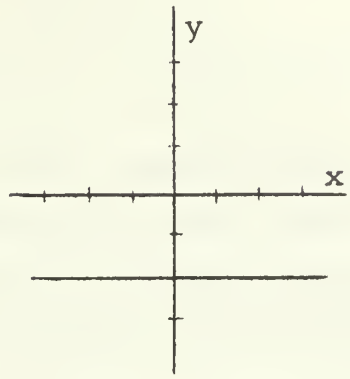


Fig. 2.

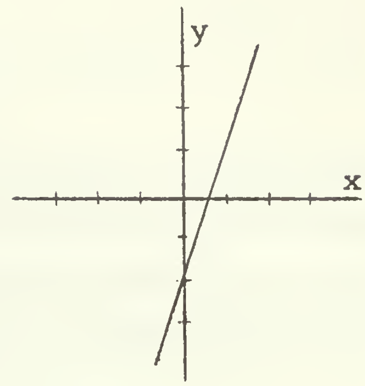


Fig. 3.

You have seen in the Exploration Exercises that a graph of each of the functions $\{(x, y): y = ax\}$, for any a, is a straight line containing the graph of the origin. Also, a graph of each of the functions $\{(x, y): y = b\}$, for any b, is a horizontal line. Now, consider graphs of the functions $\{(x, y): y = ax + b\}$, for all a and b. Are any of these graphs horizontal lines? For what values of 'a' and of 'b' will you get a horizontal line? Are any of these graphs vertical lines, that is, lines perpendicular to the graph of the x-axis? Are any of these graphs oblique straight lines, that is, lines which are neither horizontal nor vertical? For what values of 'a' and of 'b' will you get an oblique straight line?

Some of the functions $\{(x, y): y = ax+b\}$, for all a and b, are constant functions. These are the functions $\{(x, y): y = ax+b\}$, for $a = 0$ and any b. The others [that is, those which are not constant functions] are the functions $\{(x, y): y = ax+b\}$, for $a \neq 0$ and any b. These are called <u>linear functions</u> [actually, 'linear functions of a real variable', but we are abbreviating as we did in the case of constant functions].

Given a relation g, how can we tell whether it is a linear function? Suppose $g = \{(x, y): 2y - 6x + 7 = 0\}$. Is g a linear function? If we graphed it, we would find that the graph appeared to be a straight line. [Graphs of $\{(x, y): y = 7\}$ and $\{(x, y): x = 4\}$ are straight lines, but these are not linear functions. Explain.] Since the graph seems to be an oblique straight line, we are pretty sure that g is a linear function. But, graphing can sometimes be tedious as well as inaccurate. So, let's find an easier test.

First, we need a definition of 'linear function'.

f is a linear function [of a real variable]
if and only if
there are real numbers $a \neq 0$ and b such that
$f = \{(x, y): y = ax + b\}$.

This definition tells us, for example, that $\{(x, y): y = 7x - 5\}$ is a linear function since 7 and -5 are real numbers, and $7 \neq 0$. How does the definition rule out constant functions? What is the domain of a linear function? How does the definition rule out the function

$$\{(x, y): y = 7x - 5 \text{ and } |y| \leq 1000000\},$$

a large portion of whose graph looks like a straight line?

Now, let's attack the problem at hand. The relation g is $\{(x, y): 2y - 6x + 7 = 0\}$. To claim that g is a linear function is to claim that there are real numbers $a \neq 0$ and b such that

$$\{(x, y): 2y - 6x + 7 = 0\} = \{(x, y): y = ax + b\}.$$

And, to support the claim, all you must do is tell what these numbers are. How do you discover them? Just solve the set selector '$2y - 6x + 7 = 0$' for 'y'.

$$\begin{aligned} 2y - 6x + 7 &= 0 \\ 2y &= 6x - 7 \\ y &= 3x - \frac{7}{2} \end{aligned}$$

This last equation is equivalent to the given set selector. Therefore,

$$g = \{(x, y): y = 3x - \frac{7}{2}\}.$$

So, since there are numbers $a \neq 0$ and b [3 and $-7/2$, respectively] such that $g = \{(x, y): y = ax + b\}$, the definition assures us that the relation g is a linear function.

EXERCISES

<u>A</u>. For each exercise, tell whether the given relation f is a linear function. Justify your answer.

1. $f = \{(x, y): 5x + y = 3\}$
2. $f = \{(x, y): 2y - 3x - 5 = 0\}$
3. $f = \{(x, y): x = 3y + 7\}$
4. $f = \{(b, a): a = 5b + 11\}$
5. $f = \{(x, y): y + x = 9\}$
6. $f = \{(0, 9), (1, 10), (2, 11)\}$

7. $f = \{(x, y): 2(8 - 3x) - 9(2 - 3y) + 7 = 8(x - 5) + 2(4 - y)\}$.

8. $f = \{(x, y): x + y = 3 + y\}$

9. $f = \{(x, y): y + 2x = 2(x - 7)\}$

10. $f = \{(x, y): y = 7x\}$

11. $f = \{(x, y): y = 7x \text{ and } y \geq 0\}$

12. $f = \{(r, s): r = -s\}$

13. $f = \{(x, y) \in I \times I: y = 2x + 5\}$

14. $f = \{(x, y): y \geq 2x + 1\}$

15. $f = \{(x, y): y - 3 + 2(x - 4) = 0\}$

16. $f = \{(x, y): \frac{x}{3} + \frac{y}{5} = 1\}$

17. $f = \{(x, y): 5(x - 2) + 7(y - 3) = 0\}$

18. $f = \{(x, y): 3y + 2x - 5 = 8\}$

19. $f = \{(x, y): y = x^2 - x(3 + x)\}$

20. $f = \{(x, y): y = 2x \text{ or } y = 3x\}$

21. $f = \{(x, y): y = 2x \text{ and } y = 3x\}$

☆22. $f = \{(x, y): (2x - 1 - y)(x - 1) = 0\}$

☆23. $f = \{(x, y): (2x - 1 - y)(x^2 + 1) = 0\}$

B. Suppose $f = \{(x, y): y = 2x - 5\}$.

1. Does f have an inverse? If it does, is f^{-1} a linear function?

2. Prove that each linear function has an inverse, and that its inverse is a linear function.

3. Prove that if f and g are linear functions then so is $f \circ g$.

C. 1. Is each sum of a linear function and a constant function a linear function?

2. Is each product of a linear function by a constant function a linear function?

3. Is each sum of a linear function and a linear function a linear function? [Careful!]

☆D. Consider the relations $\{(x, y): Ax + By + C = 0\}$, for all real numbers A, B, and C.

1. Are any of these relations linear functions?

2. What kind of functions do we have in case $A = 0$ and $B \neq 0$? In case $B = 0$ and $A \neq 0$? In case $C = 0$ and $A \neq 0$ and $B \neq 0$? In case both A and B are 0 and $C \neq 0$? In case A, B, and C are 0?

3. Prove that, for all A, B, and C, $\{(x, y): Ax + By + C = 0\}$ is a linear function if and only if neither A nor B is 0. [Hint. Use the definition of a linear function.]

GRAPHING A LINEAR FUNCTION

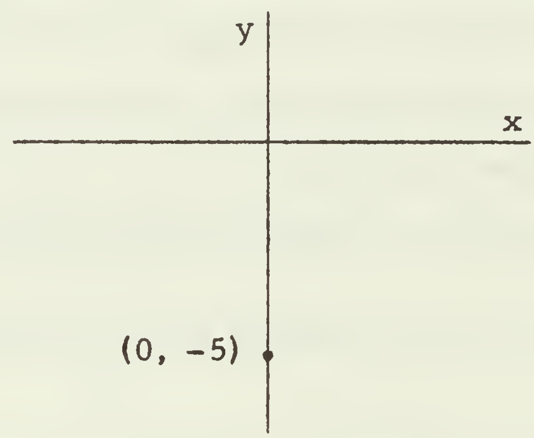
The fact that for each linear function there is a defining equation in 'y' and 'x' of the form:

$$(*) \qquad y = ax + b$$

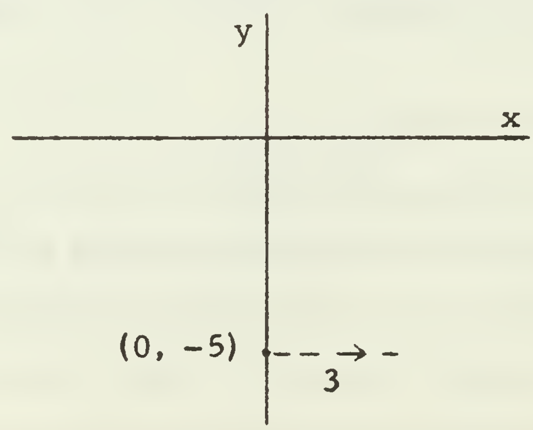
gives us a very quick way of graphing linear functions. For example, consider the linear function g where g is $\{(x, y): 2x - y - 5 = 0\}$. Since g is a linear function, we can transform the set selector to an equation of the form (*):

$$y = 2x - 5$$

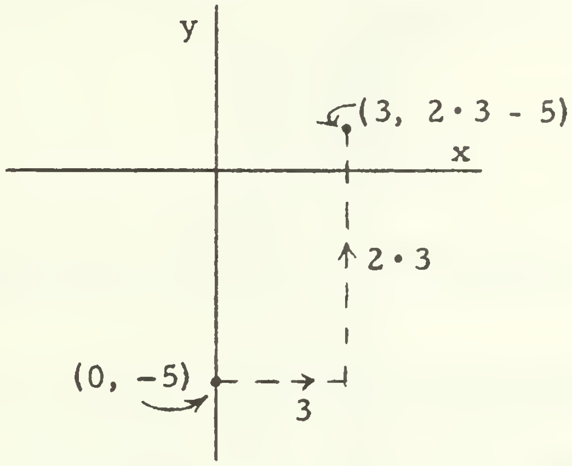
Since the domain of any linear function is the set of real numbers, we know that g is defined at 0. The defining equation '$y = 2x - 5$' tells us that $(0, -5)$ is a member of g. The graph of $(0, -5)$ is on the graph of the y-axis.



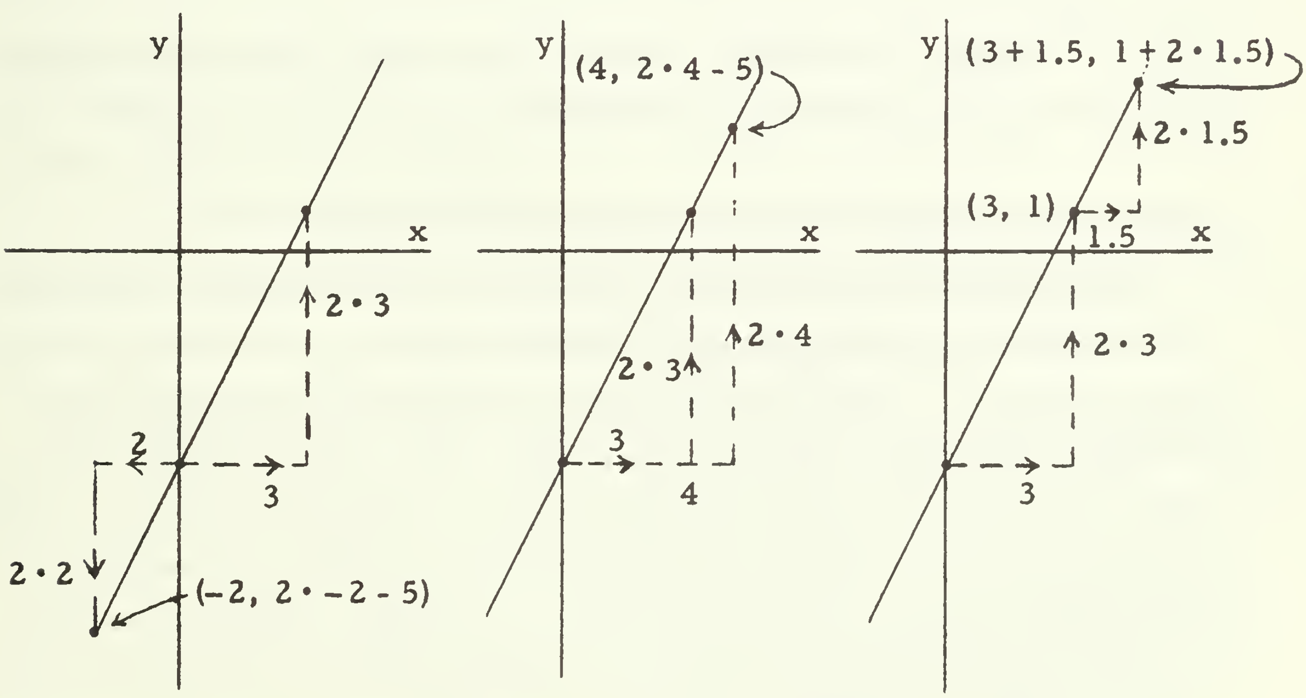
To find the graph of another member of g, place your pencil point on the graph of $(0, -5)$, and move the pencil a certain distance to the right. Say you move it a distance 3 to the right. [This brings you to the graph of $(3, -5)$, and $(3, -5)$ is <u>not</u> a member of g. How do you know that it isn't?



Now, since g is defined at 3, the vertical line through the graph of (3, −5) must contain the graph of a member of g. The x-coordinate of this graph is 3, and we use the defining equation to tell us the y-coordinate. It is $2 \cdot 3 - 5$. You reach the graph of $(3, 2 \cdot 3 - 5)$ by moving your pencil point up the distance $2 \cdot 3$.

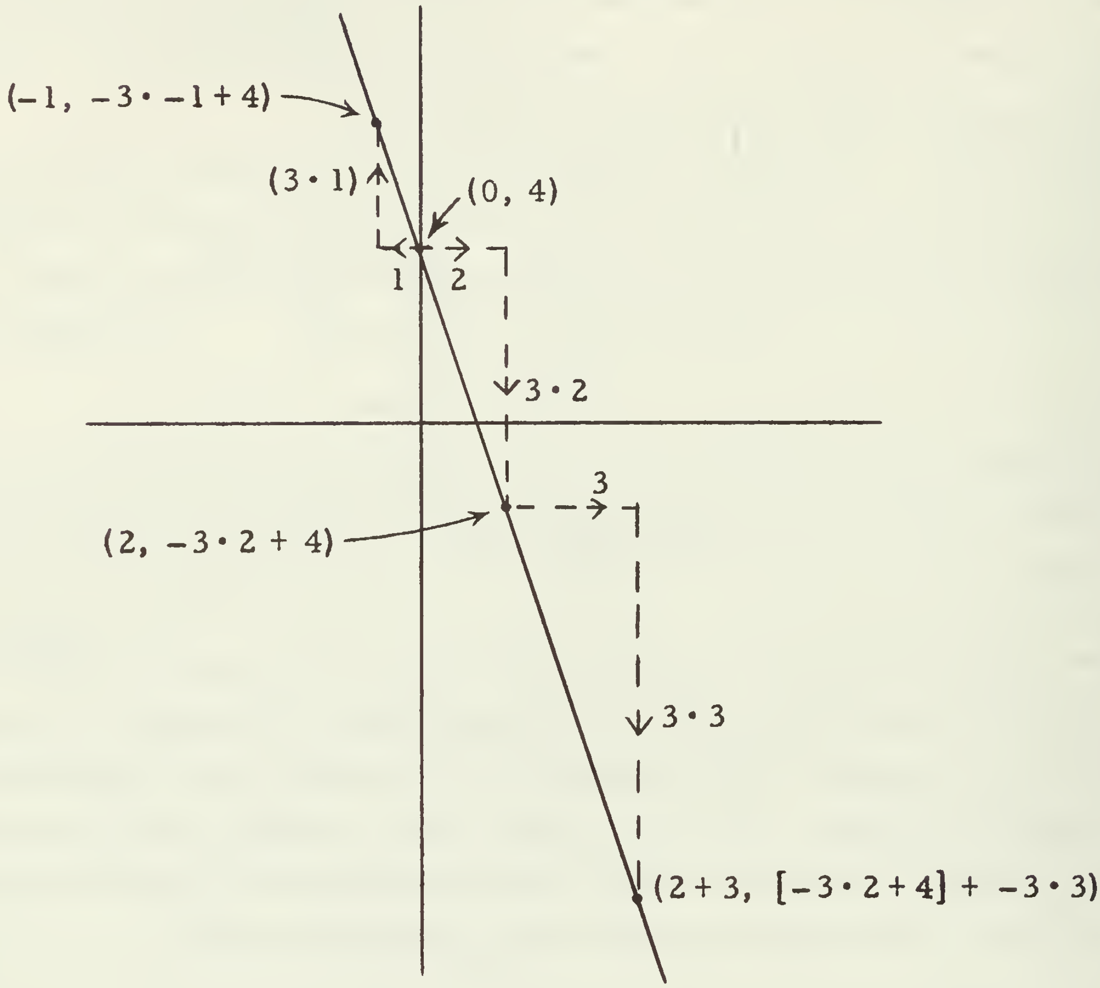


The graphs of these two members of g are all you need in order to draw the straight line which is the graph of g. However, it is good practice to graph a third member of g to catch possible plotting or computing errors. There are many ways in which you can use the ordered pairs already plotted to find the graph of a third member of g. Three such ways are shown in the diagrams below.



[Before reading any further, graph the function defined by the equation '$y = 3x - 2$' using the method illustrated above.]

Here is a diagram showing how the graphs of three more members of $\{(x, y): y = -3x + 4\}$ were obtained after graphing (0, 4).



[Study the diagram carefully, and then apply what you have learned to the job of graphing the function defined by the equation '$y = -2x + 4$'.]

SLOPE AND INTERCEPT OF A LINEAR FUNCTION

In graphing linear functions you must have noticed the important roles played by the values of 'a' and of 'b' given by the defining equation '$y = ax + b$'. The value of 'b' tells you where the graph of the function crosses the graph of the y-axis. The crossing point is the graph of what ordered pair? What does the value of 'a' tell you? Well, suppose you have already graphed one of the members of the function. Usually this is (0, b), especially if b is an integer, but let's say that you have graphed (p, q), and that you want to proceed from there to the graph of another member of the function. Move your pencil point from the graph of (p, q) along the horizontal grid line which contains the

graph of (p, q). Suppose you move it to the graph of (p + k, q). [In which direction (right or left) do you move if k is positive? If k is negative? Explain.] Now, in which vertical direction (up or down) and how far should you move to plot the graph of the member of the function whose first component is p + k? Use the defining equation to find out. Substitute 'p + k' for 'x' in the defining equation:

$$\begin{aligned} y &= a(p + k) + b \\ &= ap + ak + b \\ &= (ap + b) + ak \end{aligned}$$

Since (p, q) belongs to the function, we know that $q = ap + b$. Thus,

$$y = q + ak.$$

So, (p + k, q + ak) belongs to the function, and this is the next point to be plotted. Your pencil point is already on the graph of (p + k, q). You should move it vertically a distance $|ak|$. But, in what direction? If ak is positive, we move up because $q + ak > q$. If ak is negative, we move down because then $q + ak < q$.

If a is positive and k is positive, the graph of (p + k, q + ak) is to the right and above the graph of (p, q). In that case, we say that the graph of the function rises to the right. If a is positive and k is negative, where is the graph of (p + k, q + ak) with respect to the graph of (p, q)? Does the graph of the function rise to the right?

Suppose a is negative. Under this condition, how does the graph rise? Explain your answer.

For a given linear function f, the number b given by the defining equation tells you where the graph of f crosses the graph of the y-axis.

b is called <u>the intercept of f.</u>

Where, with respect to the horizontal axis (above or below), does the graph of f cross the vertical axis when the intercept is positive? When the intercept is negative? When the intercept is 0?

The number a given by the defining equation for f tells you how to go from the graph of one member of f to the graph of another. In doing so, it tells you how the graph rises (to the right or to the left) and whether the rise is rapid or slow.

a is called <u>the slope of f.</u>

What is the direction of rise of the graph of f if the slope is positive? If the slope is negative? [According to our definition of a linear function, can the slope be 0?] If the slope is a number close to 0 such as $-1/5$ or $2/15$, the rise is slow ["a gentle slope"]. If the slope is a number far from 0 such as 10 or -20, the rise is rapid ["a steep slope"].

Figure 1 below shows graphs of several linear functions which have the same slope but which differ in their intercepts. Figure 2 shows graphs of linear functions with the same intercept but with different slopes.

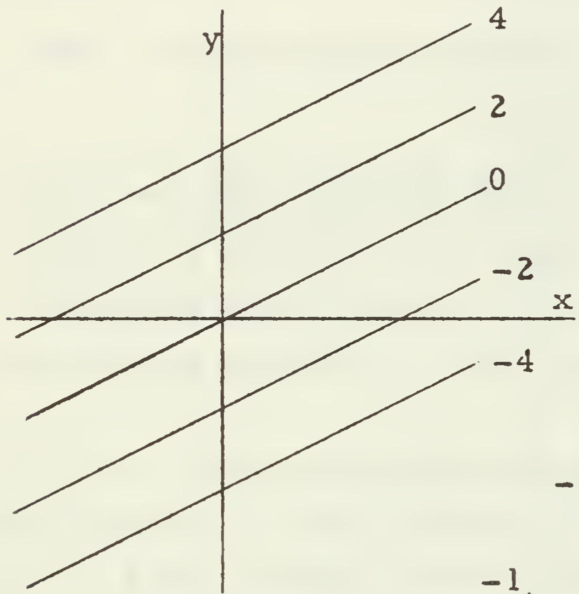


Fig. 1.

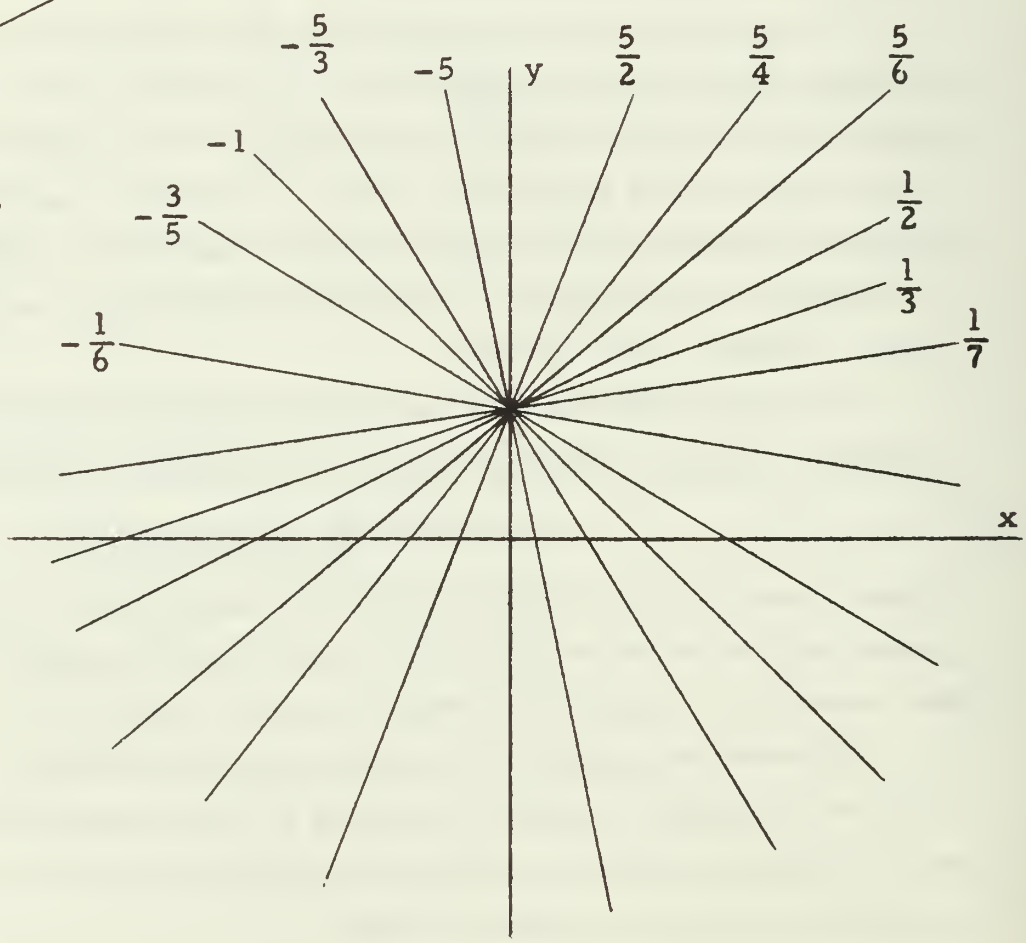
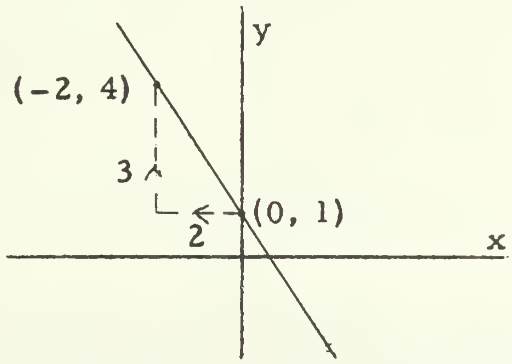


Fig. 2.

EXERCISES

A. Each of the following exercises contains a graph of a linear function. In each exercise, find the slope and the intercept of the function, and write its defining equation. [Check your work by seeing if the coordinates of the two given points satisfy the defining equation.]

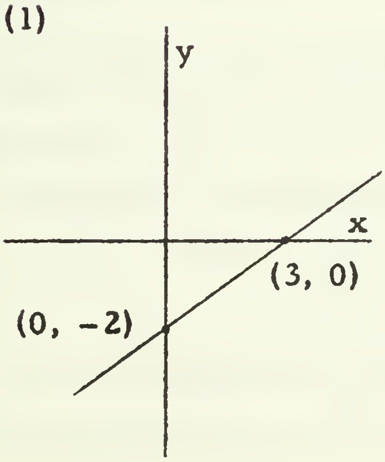
Sample.



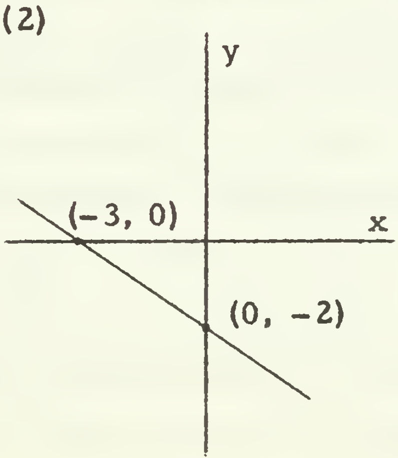
Solution. The graph rises to the left; so, the slope is negative. To get from the graph of (0, 1) to the graph of (−2, 4), you move a distance 2 in the horizontal direction and a distance 3 in the vertical direction. So, to get the vertical move you multiply the horizontal move by 3/2. Hence, the slope is −3/2. The intercept is the second component of the point in which the function intersects the y-axis. This point is (0, 1); so, the intercept is 1. The defining equation is: $y = -\frac{3}{2}x + 1$

Check.	
$4 = -\frac{3}{2} \cdot -2 + 1$?	$1 = -\frac{3}{2} \cdot 0 + 1$?
$3 + 1$	$0 + 1$
4 ✓	1 ✓

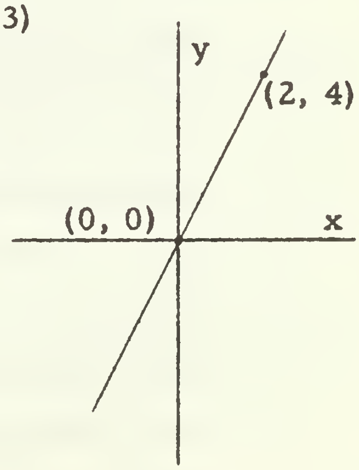
(1)



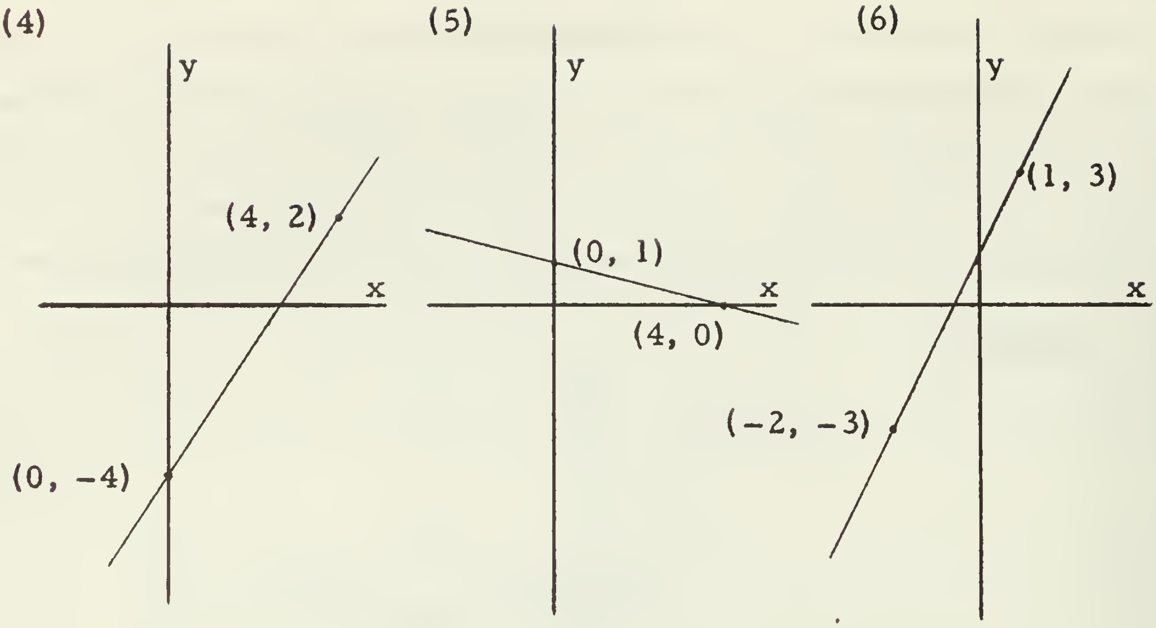
(2)



(3)



(4) (5) (6)



B. In each of the following exercises you are given the intercept and the slope of a linear function. Draw its graph and write its defining equation.

1. intercept, 3; slope, 2
2. intercept, -2; slope, $\frac{1}{3}$
3. intercept, 4; slope, $-\frac{1}{5}$
4. intercept, -1; slope, $\frac{2}{5}$

C. Is there a linear function whose slope is 0? If you were to define 'slope' for constant functions, how would you define it? How about 'intercept' for constant functions?

D. 1. Draw graphs of two linear functions with the same slope, say, $\{(x, y): y = 2x + 3\}$ and $\{(x, y): y = 2x - 4\}$. What is the set of ordered pairs which is the intersection of these functions?

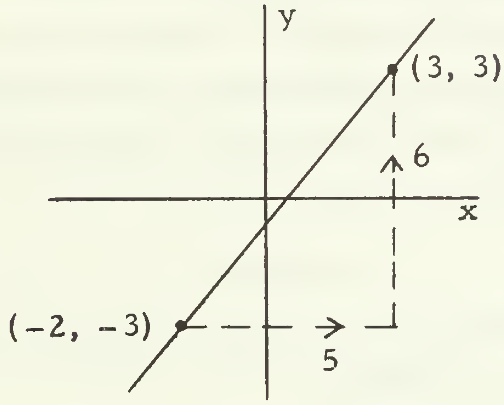
☆ 2. Prove that the intersection of two linear functions with the same slope is the empty set. [Hint. Suppose f_1 and f_2 are linear functions and that their defining equations are '$y = a_1x + b_1$' and '$y = a_2x + b_2$'. Now, each ordered pair in $f_1 \cap f_2$ satisfies both of the defining equations. Suppose (p, q) is such an ordered pair. Then $q = a_1p + b_1$ and $q = a_2p + b_2$. That is, $a_1p + b_1 = a_2p + b_2$. Or, $(a_1 - a_2)p = b_2 - b_1$, ...]

3. Show that if f_1 is a linear function and f_2 is a linear function and f_1 and f_2 have the same slope, it is not necessarily the case that $f_1 \cap f_2 = \emptyset$.

E. In each of the following exercises you are given two points. Find the slope of a linear function [if there is one] which contains these points.

Sample. $(-2, -3), (3, 3)$

Solution. Let's graph the points and draw the straight line which is determined by the graphs.



We note that the graph rises to the right. So, the slope is positive. To get from $(-2, -3)$ to $(3, 3)$, we move a distance 5 horizontally, and a distance $\frac{6}{5} \cdot 5$ vertically. So, the slope is $\frac{6}{5}$.

1. $(3, 4), (5, 8)$
2. $(0, 0), (3, 4)$
3. $(0, 0), (4, 3)$
4. $(2, 1), (-3, 8)$
5. $(4, 5), (5, 4)$
6. $(0, 4), (4, 0)$
7. $(\frac{1}{2}, 3), (-6, 5)$
8. $(3, 8), (4, 8)$
9. $(3, 5), (3, 6)$

☆ 10. $(x_1, y_1), (x_2, y_2), [x_1 \neq x_2, y_1 \neq y_2]$

F. 1. What is the slope of a linear function which contains the points $(1, 2)$ and $(3, 6)$?

2. What is the slope of a linear function which contains the points $(3, 6)$ and $(7, 14)$?

3. What is the slope of a linear function which contains the points $(2, 2)$ and $(4, 6)$?

4. Is there a linear function which contains both the pair of points mentioned in Exercise 1 and the pair mentioned in Exercise 2? Both the pair mentioned in Exercise 1 and the pair mentioned in Exercise 3?

* * *

Given the point (5, 7), how many constant functions contain this point? From the graphical point of view, this is like asking how many horizontal lines contain the graph of (5, 7). Clearly, there is just one constant function which contains the graph of (5, 7). What is it?

How many linear functions contain the point (5, 7)? Again, from the point of view of graphs, infinitely many oblique lines pass through the graph of (5, 7). [There are just as many as there are nonzero real numbers. Explain.] So, one would expect that infinitely many linear functions contain (5, 7). Let's find a few. Each linear function is defined by an equation of the form:

$$y = ax + b$$

So, a linear function contains (5, 7) if and only if the values of 'a' and 'b' in its defining equation satisfy:

$$(*) \qquad 7 = a5 + b \text{ and } a \neq 0$$

There are many pairs of numbers (a, b) which satisfy (*). For example, (1, 2) does. So does (3, −8). And, so does (80, −393). Thus, some of the linear functions which contain (5, 7) are

$\{(x, y): y = x + 2\}$, $\{(x, y): y = 3x - 8\}$, and $\{(x, y): y = 80x - 393\}$.

How many linear functions contain the points (5, 7) and (3, 11)? From our knowledge of graphs we would say 'just one'. Let's find it. This time we seek values of 'a' and 'b' which satisfy both:

$$(1) \qquad 7 = a5 + b$$

and: $$(2) \qquad 11 = a3 + b$$

We could find such values by graphing (1) and (2) and estimating the coordinates of the graph of the point in the intersection of the solution sets in (a, b) of (1) and (2). But, there are easier ways. [Before reading any further, try to discover at least one of these easier ways.]

Suppose there is a linear function which contains (5, 7) and (3, 11). From (1) we learn that the intercept of this function is 7 - a5, and from (2) we learn that the intercept is 11 - a3. Since a linear function has

exactly one intercept [How do you know this?], it must be the case that

(3) $$7 - a5 = 11 - a3.$$

From (3) we find that [if there is a linear function which contains (5, 7) and (3, 11)] the slope is −2. But, for a = −2, both (1) and (2) are satisfied by the same value, 17, of 'b'. So, there <u>is</u> a linear function which contains (5, 7) and (3, 11). It is the function

$$\{(x, y): y = -2x + 17\}.$$

Let's consider another way of finding numbers a and b which satisfy: (1) $7 = a5 + b$ and (2) $11 = a3 + b$. Suppose there are such numbers. Then, using the principle that

$$\forall_x \forall_y \forall_u \forall_v \text{ if } x = y \text{ and } u = v \text{ then } x - u = y - v,$$

it follows that these numbers a and b satisfy:

(3′) $$7 - 11 = (a5 + b) - (a3 + b),$$

that is, they satisfy: $-4 = 2a$. In other words, if there are numbers a and b such that (1) and (2) then a is −2. But, −2 satisfies (1) and (2) just if b = 17. So, there are numbers a and b which satisfy (1) and (2), and they are −2 and 17, respectively.

There is a third way of solving this problem. We find the possible values of 'a' as before. Then, we multiply to transform (1) and (2) to:

(1′) $21 = a15 + 3b$ and (2′) $55 = a15 + 5b$

Next, we use the principle stated above to conclude that if there are numbers a and b such that (1′) and (2′) [or, equivalently, such that (1) and (2)] then these numbers satisfy:

(3″) $$21 - 55 = (a15 + 3b) - (a15 + 5b),$$

that is, they satisfy: $-34 = -2b$. So, if there are numbers a and b which satisfy (1) and (2) they must be −2 and 17, respectively. Substituting in (1) and (2) shows that they do.

The methods illustrated on the preceding pages are methods for solving <u>a system of two equations in two variables</u>. You will learn more about such methods later in this unit.

* * *

G. Each exercise lists two ordered pairs of numbers. Find the linear function [if there is one] which contains them.

Sample. (2, 5), (−3, 9)

Solution.

$$\begin{aligned} 5 &= 2a + b \\ 9 &= -3a + b \\ \hline -4 &= 5a \\ -\frac{4}{5} &= a \\ 5 &= 2 \cdot -\frac{4}{5} + b \\ \frac{33}{5} &= b. \end{aligned}$$

$$\{(x, y): y = -\frac{4}{5}x + \frac{33}{5}\}$$

1. (7, 2), (13, 5)
2. (8, 6), (0, 0)
3. (4, 1), (−3, 8)
4. (2, 11), (5, 20)
5. (1, −10), (−3, 10)
6. $(\frac{1}{2}, 6)$, (3, 36)
7. (6, 9), (8, 9)
8. (0, 5), (5, 0)
9. (4, 0), (0, 3)
10. (−4, 0), (0, 3)
11. (3, 5), (3, 7)

☆12. (x_1, y_1), (x_2, y_2)

H. For each exercise, find the intersection of the given linear functions without drawing their graphs.

Sample. $\{(x, y): y = 3x + 7\}$, $\{(x, y): y = 2x + 5\}$

Solution. We are looking for numbers x and y which satisfy both:

(1) $y = 3x + 7$

and: (2) $y = 2x + 5$.

This problem can be solved by methods used in Part G. For example, if the intersection is not $\emptyset$ then there are numbers x and y which satisfy (1) and (2). Such numbers must satisfy: $0 = (3x + 7) - (2x + 5)$. Etc.
Or, if the intersection is not empty, it contains an ordered pair (p, q) such that $q = 3p + 7$ and $q = 2p + 5$. Hence, $3p + 7 = 2p + 5$. Etc.

1. $\{(x, y): y = 4x - 2\}$, $\{(x, y): y = 3x + 8\}$

2. $\{(x, y): y = -3x - 7\}$, $\{(x, y): y = 4x\}$
3. $\{(x, y): y = \frac{x - 2}{8}\}$, $\{(x, y): y = \frac{2x - 30}{3}\}$
4. $\{(x, y): y = 2 + 5x\}$, $\{(x, y): y = 5x + 1\}$
5. $\{(x, y): y - 3x = 6\}$, $\{(x, y): y - 2x = 3\}$
6. $\{(x, y): 3x + y = 4\}$, $\{(x, y): y = 2x - 1\}$
7. $\{(x, y): 3y - 6x = 9\}$, $\{(x, y): y = x + 4\}$
8. $\{(x, y): 3x - 5y = 8\}$, $\{(x, y): 8y - x = 10\}$
9. $\{(x, y): 5y + 4x = 33\}$, $\{(x, y): 3y - 7x = 48\}$
10. $\{(x, y): 4y - 3x = 12\}$, $\{(x, y): 6x = 15 + 8y\}$
11. $\{(x, y): 2y + x = 9\}$, $\{(x, y): 3y + x = 18 - y - x\}$

I. Each exercise lists three ordered pairs of numbers. Find the linear function [if there is one] which contains them.

1. $(2, 7)$, $(5, 11)$, $(8, 15)$
2. $(3, 5)$, $(8, 4)$, $(14, 2)$

*

3. Predict whether a straight line can be drawn through the graphs of $(-89, -93)$, $(67, -8)$, and $(15, -37)$. If your prediction is that it cannot, change a component of just one of the ordered pairs so that a straight line can be drawn through their graphs.

J. Each exercise lists a set of ordered pairs. In each case, tell whether there is a linear function of which the given set is a subset.

1. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
2. $\{(0, 2), (1, 4), (2, 6), (3, 8)\}$
3. $\{(-3, 7), (1, 9), (5, 11), (9, 13)\}$
4. $\{(3, 3), (2, 4), (0, 6), (-5, 11)\}$
5. $\{(1, 8), (2, 11), (3, 13), (4, 16)\}$
6. $\{(10, -1), (15, -2), (25, -4), (30, -5)\}$
7. $\{(2, \frac{1}{2}), (3, \frac{1}{3}), (4, \frac{1}{4}), (5, \frac{1}{5})\}$
8. $\{(4, 6), (5, 8), (7, 12), (9, 8)\}$

* * *

You will recall from the definition on page 5-91 that a variable quantity h is a function of a variable quantity g if and only if there is a function f such that $h = f \circ g$. We say that h is a <u>linear</u> <u>function</u> of g if one such function f is a linear function. Since the domain of each linear function f is the set of <u>all</u> real numbers, if h is a linear function of g and $\mathcal{R}_g$ consists only of real numbers then $\mathcal{D}_h = \mathcal{D}_g$. So, if g is a real-valued function, h is a linear function of g if and only if $\mathcal{D}_h = \mathcal{D}_g$ and there are real numbers $a \neq 0$ and b such that, for each $e \in \mathcal{D}_g$, $h(e) = ag(e) + b$. Another way of putting it is to say that h is a linear function of a real-valued function g if and only if $\mathcal{D}_h = \mathcal{D}_g$ and the set of all ordered pairs $(g(e), h(e))$, for $e \in \mathcal{D}_g$, is a subset of some linear function.

<u>Example</u>. Suppose $g = \{(1, 12), (2, 15), (3, 18), (4, 17)\}$
and $h = \{(1, 25), (2, 31), (3, 37), (4, 35)\}$.
Is h a linear function of g?

Since the set of ordered pairs $(g(e), h(e))$, for all $e \in \mathcal{D}_g$, is

$$\{(12, 25), (15, 31), (18, 37), (17, 35)\}$$

the question really asks whether there is a linear function of which this set of ordered pairs is a subset. One way of getting a clue to the answer is to see whether an oblique straight line can be drawn through the graphs of these ordered pairs. Another way which does not involve the inaccuracies of graphing is to try to find numbers $a \neq 0$ and b such that all four ordered pairs satisfy the equation:

$$y = ax + b.$$

If you are successful, you will not only know that h is a linear function of g but you will also have a defining equation for the linear function in question. [If you can prove that there are no numbers $a \neq 0$ and b which fit the conditions then you will know that h is not a linear function of g. Would that mean that h is not a function of g?]

There is another way of answering this question which does not require finding the linear function, itself. To understand this other way, we need to look at an important property of linear functions.

Suppose f is a linear function which contains the two points (x_1, y_1) and (x_2, y_2). [Since these are two points, and since f is a function, it

follows that $x_1 \neq x_2$. Since f is a linear function, and since $x_1 \neq x_2$, it follows that $y_1 \neq y_2$.] Now, if a and b are the slope and intercept of f, then

$$y_2 = ax_2 + b.$$

and

$$y_1 = ax_1 + b.$$

From this it follows that

$$\begin{aligned} y_2 - y_1 &= (ax_2 + b) - (ax_1 + b) \\ &= ax_2 - ax_1 \\ &= a(x_2 - x_1). \end{aligned}$$

So,

$$\frac{y_2 - y_1}{x_2 - x_1} = a.$$

In other words, if a linear function f contains the two points (x_1, y_1) and (x_2, y_2) then

$$\frac{y_2 - y_1}{x_2 - x_1} = \text{the slope of f.}$$

[Did you discover this result when you worked the exercises in Part E?]

Now, suppose (x_3, y_3) is a third point such that

$$\frac{y_3 - y_2}{x_3 - x_2} = \text{the slope of f.}$$

Does it follow that $(x_3, y_3) \in f$? The answer is 'yes'. Let's see why.

$$\frac{y_3 - y_2}{x_3 - x_2} = a$$

if and only if $\quad y_3 - y_2 = a(x_3 - x_2). \quad [x_3 \neq x_2]$

that is, if and only if $\quad y_3 = a(x_3 - x_2) + y_2.$

Since $(x_2, y_2) \in f$, we know that $y_2 = ax_2 + b$. So,

$$y_3 = a(x_3 - x_2) + y_2$$

if and only if

$$\begin{aligned} y_3 &= a(x_3 - x_2) + (ax_2 + b) \\ &= ax_3 - ax_2 + ax_2 + b \\ &= ax_3 + b, \end{aligned}$$

that is, if and only if $\quad (x_3, y_3) \in f.$

So, we have shown not only that $(x_3, y_3) \in f$ if $\frac{y_3 - y_2}{x_3 - x_2} = a$, but also that $(x_3, y_3) \in f$ <u>only</u> $\frac{y_3 - y_2}{x_3 - x_2} = a$.

Suppose we know that $\frac{y_5 - y_4}{x_5 - x_4}$ = the slope of f. Does it follow that (x_4, y_4) and (x_5, y_5) belong to f? Explain.]

We have shown that if f is a linear function and (x_1, y_1) and (x_2, y_2) are two ordered pairs such that $\{(x_1, y_1), (x_2, y_2)\} \subseteq f$, a third ordered pair $(x_3, y_3) \in f$ $[x_1 \neq x_2 \neq x_3]$ if and only if

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

This result provides us with a quick test for determining whether [or not] a given set of ordered pairs is a subset of a linear function. Let's take the example we started with on page 5-134. Is there a linear function f such that

$$\{(12, 25), (15, 31), (18, 37), (17, 35)\} \subseteq f?$$

Compute the y-difference and the x-difference for the first two ordered pairs, (12, 25) and (15, 31).

$$y_2 - y_1 = 31 - 25 = 6$$

$$x_2 - x_1 = 15 - 12 = 3$$

Then compute the y-difference and the x-difference for the second and third ordered pairs, (15, 31) and (18, 37).

$$y_3 - y_2 = 37 - 31 = 6$$

$$x_3 - x_2 = 18 - 15 = 3$$

Since the ratio of the y-difference to the x-difference in the first case is the same as the ratio of the y-difference to the x-difference in the second case, we know that there is a linear function, say, p, which contains the first three ordered pairs. Now, we find the <u>difference-ratio</u> for the third and fourth ordered pairs.

$$y_4 - y_3 = 35 - 37 = -2$$

$$x_4 - x_3 = 17 - 18 = -1$$

$$\frac{y_4 - y_3}{x_4 - x_3} = \frac{-2}{-1}$$

Is the difference-ratio here the same as the one in the case of the second and third ordered pairs? In other words, does $6/3 = -2/-1$? Yes! So, there is a linear function, say, q, which contains the second third, and fourth ordered pairs, But, there is only one linear function

which contains the second and third ordered pairs. Hence, p = q, and there is a linear function of which the given set of four pairs is a subset.

* * *

<u>K.</u> Each exercise gives a variable quantity g and a variable quantity h. In each case, tell whether h is a linear function of g.

<u>Sample 1.</u> $g = \{(a, 5), (b, -2), (c, -9), (d, -10), (e, -13)\}$
$h = \{(a, 18), (b, 4), (c, -10), (d, -12), (e, -18)\}$

<u>Solution.</u> $\mathcal{D}_h = \mathcal{D}_g$. So, far, so good. Now, consider the set of ordered pairs of corresponding values of h and g, and see whether this set is a subset of a linear function. Let's list the ordered pairs of corresponding values in a table and then compute the g(e)-differences and the h(e)-differences.

	g(e)	h(e)	
	5	18	
−7			−14
	−2	4	
−7			−14
	−9	−10	
−1			−2
	−10	−12	
−3			−6
	−13	−18	

Now, compute the difference-ratio for each pair of ordered pairs. It is 2 in each case. So, the set of ordered pairs (g(e), h(e)), for all $e \in \mathcal{D}_g$, is a subset of a linear function. Thus, h is a linear function of g.

[g is also a linear function of h because the difference-ratio is the same for the pairs of ordered pairs (h(e), g(e)). What is the difference-ratio in this case?]

1. $g = \{(\text{Al}, 9), (\text{Bill}, 4), (\text{Carl}, 6), (\text{Dora}, 12)\}$,
   $h = \{(\text{Al}, 40), (\text{Bill}, 15), (\text{Carl}, 25), (\text{Dora}, 55)\}$.

2. $g = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7)\}$,
   $h = \{(1, 1), (2, -1), (3, -3), (4, -5), (5, -7)\}$.

3. $g = \{(1, 7), (2, 5), (3, 8), (4, 9)\}$,
   $h = \{(1, 22), (2, 16), (3, 26), (4, 28)\}$.

4. $g = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$,
   $h = \{(1, 4), (2, 9), (3, 16), (4, 25)\}$.

5. $g = \{(x, y): y = x^2 + 3\}$
   $h = \{(x, y): y = x^2\}$

6. $g = \{(x, y): y = x + 3\}$
   $h = \{(x, y): y = x^2\}$

<u>L</u>. You have seen [page 5-93] that if g has an inverse and $\mathcal{D}_h \subseteq \mathcal{D}_g$ then there is a function f such that $h = f \circ g$. In fact, $h \circ g^{-1}$ is one such function. Now, if g and h are linear functions, g has an inverse, $\mathcal{D}_h = \mathcal{D}_g$, and $h \circ g^{-1}$ is a linear function. [Explain.] Hence:

If g and h are linear functions then h is a linear function of g.

For each of Exercises 1-4, find a linear function f such that $h = f \circ g$.

1. $g = \{(x, y): y = x - 7\}$
   $h = \{(x, y): y = 5 - 2x\}$

2. $g = \{(x, y): x + 3y = 6\}$
   $h = \{(x, y): 3x - 4y = 5\}$

3. $g = \{(x, y): y = 3x - 4\}$
   $h = \{(x, y): x = 5y + 2\}$

4. $g = \{(x, y): y = 5x - 6\}$
   $h = \{(x, y): 2x + 3y = 7\}$

*

5. If f is a function which has an inverse then

$$f^{-1} \circ f = \{(x, y) \in \mathcal{D}_f \times \mathcal{D}_f : y = x\}.$$

Explain the remark just made, and use it in proving that if a function h is a linear function of a real-valued function g, then g is a linear function of h.

* * *

Strictly speaking [as, up to now, we have been], since the values of each linear function are real numbers, a function h which is a linear function of a function g must also be real-valued. However, it is convenient to say that a function h whose values are numbers of arithmetic is a linear function of another such function g when $\mathcal{D}_h = \mathcal{D}_g$ and there exist numbers $a \neq 0$ and b such that, for each $e \in \mathcal{D}_g$, $h(e) = ag(e) + b$. This amounts to pretending that each value of h or g is the corresponding nonnegative real number.

* * *

<u>M</u>. 1. (a) Is the circumference of a circle a linear function of its diameter?

(b) If it is, find the linear function in question.

2. (a) Is the area-measure of a square a linear function of its side-measure?

(b) If it is, find the linear function in question.

(c) Is the area-measure of a square a linear function of the <u>square</u> of its side-measure?

3. The table in each exercise lists the pairs of corresponding values of the variable quantities A and B. Write a formula for A in terms of B.

(a)

B(e)	A(e)
3	5
4	7
5	9
6	11
7	13

(b)

B(e)	A(e)
−2	−11
0	−1
2	9
4	19
6	29

(c)

A(e)	B(e)
5	16
7	22
10	31
14	43
19	58

(d)

B(e)	A(e)
12	−12
−15	15
−12	12
15	−15
0	0

(e)

A(e)	B(e)
10	56
9	51
8	46
7	41
6	36

(f)

A(e)	B(e)
0.1	6.5
0.2	7
0.3	7.5
0.4	8
0.5	8.5

(g)

B(e)	A(e)
2	7
5	4
8	1
9	0
12	−3

(h)

B(e)	A(e)
3	9
4	16
6	36
8	64
12	144

(i)

B(e)	A(e)
3	8
4	6
6	4
8	3
12	2

[Supplementary exercises are in Part S, pages 5-267 through 5-269.]

5.08 Applications of linear functions.--One very common application of linear functions involves the notion of variable quantities which are proportional to each other. The word 'proportional' is used a great deal in everyday life.

> The city of Zabranchburg will increase its educational budget by 20% in the next school year. However, local school taxes will not increase in the same proportion because most of the increase in expenses will be taken care of by state funds.
>
> The elongation of a stretched spring is proportional to the stretching force.
>
> Most of these recipes can be used in preparing food for a larger number of persons than indicated just by increasing the ingredients a proportional amount.

The word 'proportional' has been given a very precise meaning in mathematics. Before we state a definition, investigate your present understanding of the word by reading each of the following statements and deciding whether it is true or whether it is false.

True	False	
.....	.....	1. The perimeter of a square is proportional to the measure of a side.
.....	.....	2. The weight of a boy is proportional to his height.
.....	.....	3. The volume of a gas sample under a given pressure is proportional to the temperature.
.....	.....	4. The weight of a steel beam of a given cross sectional area is proportional to the length of the beam.
.....	.....	5. The amount of juice in an orange is proportional to the radius of the orange.
.....	.....	6. The distance traveled by a car operating at a given speed is proportional to the time of travel.

.... 7. A television repairman's labor charge is proportional to the time he works.

.... 8. The compound interest on a sum of money deposited in a bank is proportional to the length of time the sum of money is left on deposit.

.... 9. The annual simple interest on a loan of a given principal is proportional to the interest rate.

.... 10. The number of years a student has been in school is proportional to the number of years in his age.

Take the first of these statements, the one which tells you that the perimeter of a square is proportional to the measure of a side. In brief, it says that the variable quantity P is proportional to the variable quantity s. According to the definition we shall soon give, this statement is true. If we double the side-measure of a square, we get a new square whose perimeter is double the perimeter of the given square. If we increase the side-measure by adding 3 to it, the perimeter increases proportionately by 12. On the other hand, the last statement in the list is false. A student who is 8 years old has been in school, say, 2 years. But, a student who is only twice as old will have been in school five times as long.

For variable quantities P and Q,
P is proportional to Q
if and only if
$\mathcal{D}_P = \mathcal{D}_Q$, and there is a number $k \neq 0$ such that,
for each $e \in \mathcal{D}_Q$, $P(e) = kQ(e)$.
[We say that P is proportional to Q with <u>factor of proportionality k</u>.]

Illustrate this definition in the case of P and s for squares. Show that P is proportional to s and tell the factor of proportionality. Also, show that s is proportional to P, and tell the factor of proportionality in that case.

Use the definition to show that the perimeter of a rectangle whose length-measure is 5 is <u>not</u> proportional to the width-measure of the rectangle.

EXERCISES

A. In each of the following exercises there is a table which lists the pairs of corresponding values of variable quantities P and Q. In each case, tell whether P is proportional to Q, and if it is, give the factor of proportionality.

Sample.

P(e)	11	8	7	5	9	6
Q(e)	22	16	14	10	18	13

Solution. We note that each value of P is one half the corresponding value of Q, except for the value 6 of P. So, P is not proportional to Q.

Answer. P is not proportional to Q because $9 = \frac{1}{2} \cdot 18$, but $6 \neq \frac{1}{2} \cdot 13$.

1.

P(e)	Q(e)
9	3
12	4
15	5
−21	−7
0	0
−9	−3

2.

Q(e)	P(e)
3	5
4	6
5	7
−7	−5
0	2
−3	−1

3.

Q(e)	P(e)
−3	9
−2	6
−1	3
0	0
1	−3
2	−6

4.

P(e)	Q(e)
4	−2
1	−1
0	0
1	1
4	2
9	3

5.

Q(e)	P(e)
3	5
4	20/3
6	10
−6	−10
0	0
−12	−20

6.

Q(e)	P(e)
11	11
6	6
−2	−2
5	5
1	1
3	3

7.

P(e)	3	3	3	3	3	3	3	3	3
Q(e)	8	8	8	8	8	8	8	8	8

8.

Q(e)	2	3	4	5	6	7	8	9	10
P(e)	6	9	12	15	18	22	24	27	30

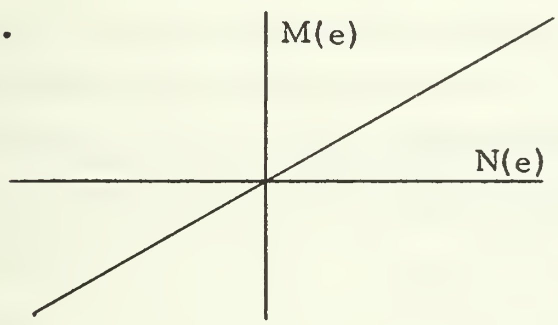
B. 1. Show that the circumference of a circle is proportional to its radius, and give the factor of proportionality.

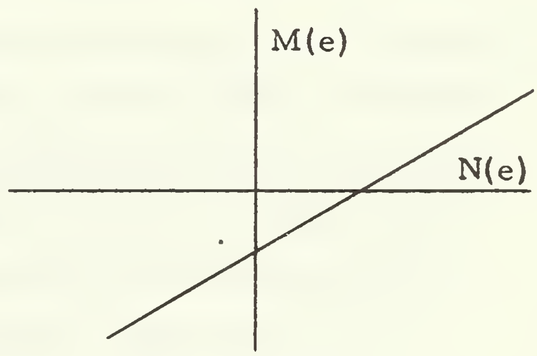
2. Is the perimeter of a regular hexagon proportional to its side-measure?

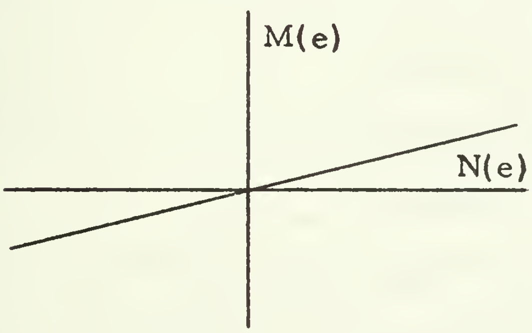
3. Show that the area-measure of a square is not proportional to its side-measure.

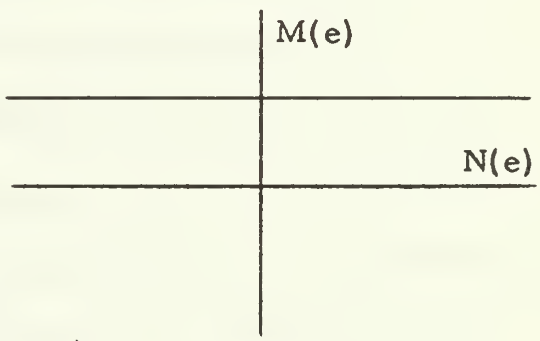
4. You know that the volume-measure of a circular cone is $\frac{1}{3}\pi r^2 h$. Show that the volume-measure of a circular cone whose base has radius-measure 3 is proportional to the height-measure of the cone, and give the factor of proportionality.

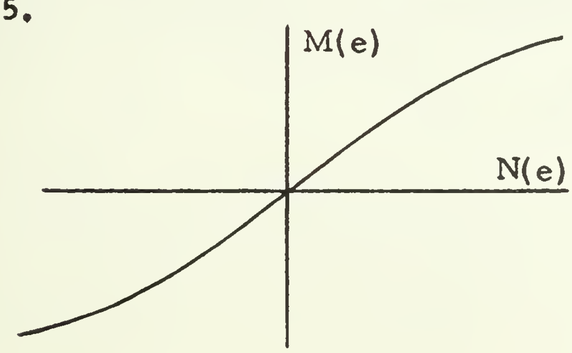
C. Here are some charts which show the graphs of sets of ordered pairs of corresponding values of variable quantities M and N. [Assume that, in each case, the entire set has been graphed.] For each exercise, tell whether M is proportional to N.

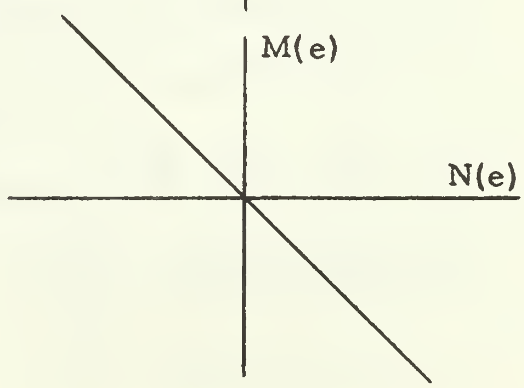
1.


2.


3.


4.


5.


6.


D. Complete these sentences.

1. If P is proportional to Q with factor of proportionality k then Q ________ ['is' or 'is not'] proportional to P. [If Q is proportional to P, what is the factor of proportionality?]

2. If P is proportional to Q then P ________ ['is' or 'is not'] a linear function of Q. [If P is a linear function of Q, what is the slope and what is the intercept of the linear function?]

3. If P is proportional to Q with factor of proportionality k then, for each $e \in \mathcal{D}_Q$ such that $Q(e) \neq 0$,

$$\frac{P(e)}{Q(e)} = \_\_\_\_\_\_ \text{ and } \frac{Q(e)}{P(e)} = \_\_\_\_\_\_.$$

4. (a) If P is proportional to Q with factor of proportionality k then, for each $e \in \mathcal{D}_Q$ such that $Q(e) \neq 0$, $[\frac{P}{Q}](e) =$ ________.

   (b) Is $\frac{P}{Q}$ a variable quantity?

E. Suppose Y is proportional to X and that none of the values of X is 0. Suppose, further, that y_1 and x_1 are corresponding values of Y and X. So, if k is the factor of proportionality then $y_1 = kx_1$. Similarly, if y_2 and x_2 are other corresponding values of Y and X then $y_2 = kx_2$.

1. If $y_1 = 8$ and $x_1 = 2$ then $k =$ ________.
2. If $y_1 = 12$ and $x_1 = 10$ then $k =$ ________.
3. If $x_2 = 3.5$ and $k = 4$ then $y_2 =$ ________.
4. If $y_2 = 360$ and $k = 12$ then $x_2 =$ ________.
5. If $y_1 = 16$, $x_1 = 4$, and $y_2 = 28$ then $x_2 =$ ________.
6. If $y_1 = 3$, $x_1 = 7$, and $x_2 = 21$ then $y_2 =$ ________.
7. If $y_2 = 35$, $y_1 = 5$, and $x_2 = 14$ then $x_1 =$ ________.
8. Prove that none of the values of Y is 0.
9. If $\frac{y_1}{x_1} = 19$, $\frac{y_2}{x_2} =$ ________.
10. Prove that $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.
11. Prove that $\frac{x_1}{y_1} = \frac{x_2}{y_2}$.

12. If $x_1 y_2 = 12$ and $x_2 = 4$ then $y_1 =$ ________.

13. Prove that $x_1 y_2 = x_2 y_1$.

14. If $\frac{x_1}{x_2} = \frac{2}{3}$ then $\frac{y_2}{y_1} =$ ________.

15. If $x_1 = 6x_2$ and $y_1 = 54$ then $y_2 =$ ________.

16. If $y_1 + x_1 = 17$, $x_1 = 3$, and $x_2 = 6$ then $\frac{y_2 + x_2}{x_2} =$ ________.

17. Prove that $\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$. [Hint. Prove, first, that $\frac{y_1}{x_1} + 1 = \frac{y_2}{x_2} + 1$.]

18. If $\frac{y_1}{x_1} = \frac{3}{8}$, $y_1 = 12$, and $x_2 = 40$ then $\frac{y_1 + y_2}{x_1 + x_2} =$ ________.

☆ 19. (a) Prove that $\frac{y_1 + y_2}{x_1 + x_2} = \frac{y_1}{x_1}$ [provided $x_1 \neq -x_2$].

(b) Prove that, for all m, n, p, and q, if (m, n) and (p, q) belong to a linear function whose intercept is 0 then so does (m + q, n + q).

PROPORTIONS

Suppose the variable quantity Y is proportional to the variable quantity X with factor of proportionality k, and that y_1 and x_1 are corresponding values and that y_2 and x_2 are other corresponding values. It is customary to say, in this case, that

(1) y_1, x_1, y_2, and x_2 are <u>in proportion</u>.

The ancient Greeks expressed (1) by writing:

(2) $$y_1 : x_1 :: y_2 : x_2.$$

[which is read as 'y_1 is to x_1 as y_2 is to x_2']. Sentence (2) is sometimes called <u>a proportion</u>. Since $y_1 = kx_1$ and $y_2 = kx_2$, it follows that if neither x_1 nor x_2 is 0 then

(3) $$\frac{y_1}{x_1} = k \text{ and } \frac{y_2}{x_2} = k.$$

This latter fact is often expressed by saying that if Y is proportional to X with factor of proportionality k then all pairs of corresponding nonzero values have the same ratio [or: a <u>fixed</u> ratio], the ratio being k if it is computed by dividing Y-values by X-values, or $\frac{1}{k}$ if it is

computed by dividing X-values by Y-values. Sometimes we eliminate the fixed ratio from consideration by deriving from (3) the equation:

$$(4) \qquad \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Sentence (4) is also called a proportion. Since we shall have no need to use sentences like (2), our use of the word 'proportion' will be restricted to sentences like (4). Thus, a proportion is an equation each of whose sides is a fraction.

Suppose Y is proportional to X and y_1, y_2, y_3, ... are values of Y which correspond with the nonzero values x_1, x_2, x_3, ... of X. Then, we can write:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}, \frac{y_2}{x_2} = \frac{y_3}{x_3}, \ldots .$$

This conjunction of proportions is usually abbreviated to:

$$(5) \qquad \frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} = \ldots$$

[Does it follow from (5) that $\frac{y_1}{x_1} = \frac{y_3}{x_3}$?]

EXERCISES

A. Solve these proportions.

Sample. $\frac{3}{a} = \frac{5}{7}$

Solution.

$$\frac{3}{a} = \frac{5}{7}$$

$$(7a)\frac{3}{a} = \frac{5}{7}(7a)$$

$$21 = 5a$$

$$4.2 = a$$

The root of the proportion is 4.2.

1. $\frac{6}{x} = \frac{2}{5}$
2. $\frac{3}{y} = \frac{7}{9}$
3. $\frac{8}{5} = \frac{16}{b}$
4. $\frac{2}{1} = \frac{x}{4}$
5. $\frac{x}{9} = \frac{5}{2}$
6. $\frac{8}{x} = \frac{1}{5}$
7. $\frac{5a}{3} = \frac{4}{9}$
8. $\frac{6}{7b} = \frac{2}{5}$
9. $\frac{4.01}{2x} = \frac{8.02}{3}$
10. $\frac{3}{x} = \frac{x}{12}$
11. $\frac{y}{2} = \frac{8}{y}$
12. $\frac{4}{z} = \frac{z}{16}$

13. $\frac{x - 5}{2} = \frac{9}{8}$ 14. $\frac{5}{3 - x} = \frac{10}{11}$ 15. $\frac{7}{3} = \frac{2x + 1}{9}$

16. $\frac{2}{x} = \frac{6}{12} = \frac{y}{18}$ 17. $\frac{x}{3} = \frac{y}{15} = \frac{18}{27}$ 18. $\frac{8}{x} = \frac{y}{4} = \frac{6}{8} = \frac{5}{z}$

B. 1. The perimeter (P) of an equilateral triangle is proportional to its side-measure (s). Suppose t_1 and t_2 are two equilateral triangles, and that $P(t_1) = P_1$, $s(t_1) = s_1$, $P(t_2) = P_2$, and $s(t_2) = s_2$. Write a proportion involving 'P_1', 's_1', 'P_2', and 's_2'.

2. Suppose the variable quantity A is proportional to the variable quantity B, and that, for some x in the domain of B, $A(x) = 10$ and $B(x) = 2$. Write a formula for A in terms of B.

3. On squared paper make drawings of five squares, compute the perimeter, and measure the diagonal of each. Plot the five pairs (d_1, P_1), (d_2, P_2), ..., (d_5, P_5). Is P proportional to d? If so, use the graph to estimate the factor of proportionality.

4. On squared paper make drawings of five rectangles each with width-measure 5. Measure the length and the diagonal, and compute the perimeter of each, recording your data in a table like this:

$\ell(r)$	$d(r)$	$P(r)$

(a) Graph the ordered pairs $(\ell(r), P(r))$, for all of the five rectangles r.

(b) Does the graph suggest that P is a linear function of ℓ?

(c) Does the graph suggest that P is proportional to ℓ? [If so, estimate the factor of proportionality.]

(d) Graph the ordered pairs $(d(r), P(r))$, for all of the five rectangles r.

(e) Does the graph suggest that P is a linear function of d? That P is proportional to d?

(f) Do you think that d is a linear function of ℓ? That d is proportional to ℓ?

5. Draw five rectangles such that the length of each is twice its width. Make measurements and computations and record the data in a table like this:

$w(r)$	$\ell(r)$	$d(r)$	$P(r)$

(a) Graph the ordered pairs $(w(r), d(r))$, for all of the five rectangles r.

(b) Does the graph suggest that d is a linear function of w? That d is proportional to w?

(c) Do you think that P is a linear function of w? That P is proportional to w? [If the latter, what is the factor of proportionality?]

(d) Do you think that P is a linear function of d? That P is proportional to d? [If the latter, what is the factor of proportionality?]

(e) Is the domain of the variable quantities in Exercise 5 the same as the domain of those in Exercise 4?

6. Imagine an automobile traveling on a highway at a constant [or: fixed] rate of 50 miles per hour for a period of 10 hours. Consider the variable quantities d and t where d is the distance traveled and t is the time which has elapsed since the beginning of the period. [You might think of the domain of d and t as a set of observations. For the first observation, o_1, $t(o_1) = 0$ and $d(o_1) = 0$. The second observation, o_2, might be made 1.5 hours later in which case $t(o_2) = 1.5$ and $d(o_2) = 75$.]

(a) Make as many observations as you wish [after all, this is only pretending]. For each observation, o, plot the ordered pair $(t(o), d(o))$. [A common way of saying this is: <u>plot d against t</u>.]

(b) Now, imagine the same situation as before except that the constant rate is 40 miles per hour. On the same chart as in (a), plot d against t.

(c) How is the difference in rates reflected in the difference in graphs?

(d) In the time it takes the faster car to travel 325 miles, how far does the slower one travel?

(e) In each case, is d proportional to t? If so, give the factor of proportionality in each case.

* * *

> Variable quantities M and N are proportional if and only if they have the same domain and there is a nonzero constant k such that $M = kN$.

Suppose that P is proportional to Q, with the factor of proportionality 3. From the formula '$P = 3Q$' [together with the fact that P and Q have the same domain], it follows that $\frac{P}{Q} = 3$, where 3 is a constant whose domain consists of those arguments for which Q has nonzero values. So, the ratio of P to Q is a constant. Evidently, then, the ratio of proportional variable quantities is a constant. Here is a more useful statement of this result:

> Variable quantities M and N are not proportional if $\frac{M}{N}$ is not a constant.

For example, suppose that A and B are variable quantities such that $A = 6B + 5$. Dividing by B, we find that

$$\frac{A}{B} = 6 + \frac{5}{B}.$$

So [unless B is itself a constant] the ratio of A to B is not a constant. Hence, supposing that B is not a constant, A is not proportional to B.

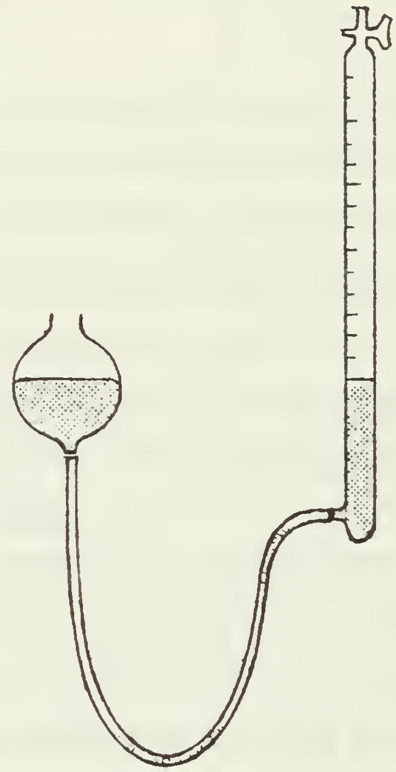
* * *

C. For each of the following formulas, tell whether the dependent variable is proportional to the independent variable. If it is, give the factor of proportionality. [Assume that both variable quantities have the same domain, and that neither is a constant.]

1. $A = 3B + 2$
2. $P = 5s$
3. $C = \pi D$
4. $Y = \pi x^2$
5. $E = 9 \times 10^{16} \times M$
6. $P = \frac{3}{Q}$

INVERSE PROPORTIONALITY

Robert Boyle, a seventeenth century English scientist, studied the effect of change of pressure on the volume of a sample of gas which is kept at a constant temperature. A type of experiment he may have carried out is illustrated in the diagram. A sample of air is confined



in a glass tube, and the pressure on this sample of air is changed by lowering or raising the bulb of mercury. It seems clear that as the pressure is increased by raising the bulb, the volume occupied by the sample of air is decreased. Boyle noticed that all gases behaved alike when subjected to changes in pressure, and his experiments gave him reason to state the following principle. [The principle is known as Boyle's law.]

> The volume occupied by a sample of gas at constant temperature is inversely proportional to the pressure.

At the top of the next page is a table which lists pairs of corresponding values of the variable quantities P and V for a sample of helium at 0° centigrade.

P	V
1.0852	20.65
1.0020	22.37
0.8067	27.78
0.6847	32.73
0.5387	41.61
0.3550	63.10
0.1937	115.65

What does 'is inversely proportional to' mean? It certainly doesn't mean what 'is proportional to' does. You can see that if you check the ratios for the first two pairs of values. Is 20.65/1.0852 the same as 22.37/1.0020? [Try to answer this question without doing any computing.] But, if, instead of considering the ratios for pairs of corresponding values, we consider the products, we find that each product is 22.41 correct to the nearest hundredth. So, Boyle's law tells us that, for a fixed temperature, the product of corresponding values of the variable quantities P and V is fixed. That is, PV is a constant. [Actually, Boyle's law does not hold at very high pressures or at very low temperatures. See an encyclopedia or a physics of chemistry textbook for more information about Boyle's law.]

We now give a definition of inverse proportionality.

For variable quantities M and N,
M is inversely proportional to N
if and only if
$\mathcal{D}_M = \mathcal{D}_N$, and there is a number $k \neq 0$ such that,
for each $e \in \mathcal{D}_N$, $M(e) \cdot N(e) = k$.

[k is the factor of inverse proportionality.]

There are many examples of variable quantities which are inversely proportional. For instance, consider the set of all rectangles r whose area-measure is 24, and the variable quantities b and h [base and height] whose domain is this set of rectangles. Complete the table below which lists just a few of the pairs of corresponding values of b and h.

b(r)	24			12		8	6	5	4	3	2	1
h(r)		1.2	1.5		2.4							

In order to complete the table, you probably used the idea that, for each rectangle r in the domain of b and h, b(r) • w(r) = 24. And, this is the same as saying that b is inversely proportional to h [and, also, that h is inversely proportional to b].

Another example of inverse proportionality is the situation involving the set of all annual loans which bear $60 in interest. The variable quantities are the principal (p) loaned and the annual interest rate (r). A principal of $1000 would require a rate of 6% to yield $60 in interest in one year. But, a principal of $2000 would require a rate of 3%. So, there is a $k \neq 0$ such that, for each loan ℓ in the domain of p and r, $p(\ell) \cdot r(\ell) = k$. In this case, $k = 60$.

You may wonder how the notion of proportion which deals with fixed ratios is tied up with that of inverse proportionality which deals with fixed products. Recall how proportions came into the picture when we discussed direct proportionality. [We shall say 'direct proportionality' now instead of just 'proportionality' since we have two kinds of proportionality to keep in mind.] We said that M is [directly] proportional to N if and only if $\mathcal{D}_M = \mathcal{D}_N$, and there is a $k \neq 0$ such that, for each $e \in \mathcal{D}_N$, $M(e) = kN(e)$. From this it follows that if m_1, n_1, and m_2, n_2 are pairs of corresponding values of M and N, then

(1) $$m_1 = kn_1 \text{ and } m_2 = kn_2.$$

From (1) [assuming that neither n_1 nor n_2 is 0] we derive the proportion:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2}$$

We can derive other proportions from (1) and the assumption about n_1 and n_2. In particular, by the principle:

$$\forall_x \forall_y \forall_{u \neq 0} \forall_{v \neq 0} \text{ if } x = y \text{ and } u = v \text{ then } \frac{x}{u} = \frac{y}{v},$$

it follows that

$$\frac{m_1}{m_2} = \frac{kn_1}{kn_2},$$

or, more simply, that

(1′) $$\frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

Now, consider the case of inverse proportionality. By definition,

M is inversely proportional to N if and only if $\mathcal{D}_M = \mathcal{D}_N$, and there is a $k \neq 0$ such that, for each $e \in \mathcal{D}_N$, $M(e) \cdot N(e) = k$. As before, if m_1, n_1, and m_2, n_2 are corresponding values of M and N [none of the values of M and N is 0; explain], then

(2) $$m_1n_1 = k \text{ and } m_2n_2 = k.$$

From this and the principle for real numbers mentioned on page 5-152 it follows that

$$\frac{m_1n_1}{m_2n_2} = \frac{k}{k}.$$

This can be transformed to:

$$\frac{m_1n_1}{m_2n_2} \cdot \frac{n_2}{n_1} = \frac{k}{k} \cdot \frac{n_2}{n_1}$$

Simplifying, we get a proportion:

(2′) $$\frac{m_1}{m_2} = \frac{n_2}{n_1}$$

Compare (1′) and (2′).

In the exercises which follow we sometimes use the phrases <u>varies directly as</u> and <u>varies inversely as</u>. These mean precisely the same things as 'is directly proportional to' and 'is inversely proportional to'. Similarly, cases of direct proportionality are sometimes referred to as cases of <u>direct variation</u>, and cases of inverse proportionality as cases of <u>inverse variation</u>.

EXERCISES

<u>A</u>. Suppose Y and X are variable quantities such that Y is inversely proportional to X. y_1, y_2, y_3, ... are values of Y which correspond with the values x_1, x_2, x_3, ... of X. Fill the blanks.

<u>Sample</u>. If $y_1 = 5$, $x_1 = 8$, and $y_2 = 4$ then $x_2 =$ ______.

<u>Solution</u>. There are several ways of handling this problem. One way is to find k, the factor of inverse proportionality. k is the product of each pair of corresponding values of Y and X. So,

$$k = y_1x_1 = 5 \cdot 8 = 40.$$

Therefore,

$$y_2x_2 = 40$$

$$4x_2 = 40$$

$$x_2 = 10.$$

A second way is to use the fact that

$$y_1x_1 = y_2x_2.$$

Substituting gives us: $5 \cdot 8 = 4x_2$

We transform to get: $x_2 = 10$

A third way is to set up a proportion:

$$\frac{y_1}{y_2} = \frac{x_2}{x_1},$$

substitute in it:

$$\frac{5}{4} = \frac{x_2}{8},$$

and then solve it.

1. If $y_1 = 8$ and $x_1 = 2$ then $k =$ ________.
2. If $y_1x_1 = 48$ and $x_2 = 3$ then $y_2 =$ ________.
3. If $y_1 = 5$, $x_1 = 9$, and $y_2 = 3$ then $x_2 =$ ________.
4. If $y_1 = 2$, $x_1 = 10$, and $x_2 = 5$ then $y_2 =$ ________.
5. If $x_1 = 3x_2$ and $y_2 = 12$ then $y_1 =$ ________.
6. If $\frac{x_1}{x_2} = \frac{3}{4}$ then $\frac{y_1}{y_2} =$ ________.
7. If $y_1 = 3.2$, $y_2 = 0.32$, and $x_1 = 2.8$ then $x_2 =$ ________.

B. 1. Turn to the table you completed on page 5-151, and plot b against h using as many more ordered pairs (h(r), b(r)) as you need to get a "smooth" graph.

2. From the appearance of your graph and from what you know about these variable quantities, would you say that b is a function of h? If so, give a function f such that $b = f \circ h$.

3. Do you think that b is a linear function of h?

☆4. If your answer to Exercise 3 is 'yes', give the linear function in question. If your answer is 'no', prove that there is no linear function g such that $b = g \circ h$.

5. Graph the function $\{(x, y): xy = 3\}$. What are the domain and range of this function? This function is called <u>an hyperbola.</u> The graph you drew in Exercise 1 is a picture of a <u>branch</u> of an hyperbola. Just as the oblique straight line through the origin is characteristic of direct proportionality [or: direct variation], so the hyperbola whose branches come arbitrarily close to the axes is characteristic of inverse proportionality [or: inverse variation].

<u>C</u>. Suppose the variable quantity A varies inversely as the variable quantity B, that their common range is the set of positive real numbers, and that, for some $e \in \mathcal{D}_B$, $A(e) = 1$ and $B(e) = 2$.

1. Write a formula for A in terms of B.

2. Plot A against B.

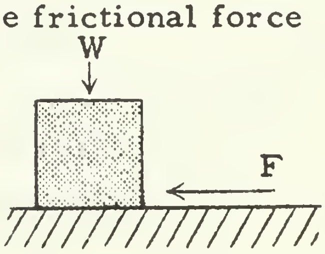
3. Plot A against the variable quantity $\frac{1}{B}$.

4. Does the graph in Exercise 3 suggest that A is a linear function of $\frac{1}{B}$? If so, what linear function?

5. Is it the case that if M varies directly as N then M varies inversely as $\frac{1}{N}$? If it is and if the factor of direct variation is k, what is the factor of inverse variation?

6. Show how each case of inverse variation can be viewed as a case of direct variation.

<u>D</u>. 1. The frictional force on a body sliding on a horizontal surface varies directly as the weight of the body. If the frictional force on a 10-lb. block of wood sliding on a horizontal surface is 3 pounds, what would be the frictional force on a 25-lb block of wood sliding on the same surface?



2. Since F varies directly as W, what is the factor of direct variation? [In the case of frictional forces and sliding bodies, the factor of direct variation is technically called the coefficient of sliding friction.]

3. Write a formula for F in terms of W, and use the formula to tell the frictional force on a 100-lb. block of wood sliding on the same horizontal surface mentioned in Exercise 1.

E. 1. A bookstore is running a sale on its nonfiction books. If the sale price (S) of a book varies directly as its list price (L), and a \$4.50-book [that is, a book whose list price is \$4.50] is on sale for \$4.05, what should be the sale price on a \$6.80-book?

2. What is the list price of a book which is on sale for \$6.93?

3. Write a formula for S in terms of L. What is the discount rate for the sale?

F. Here is a list of stations along the Illinois Central Railroad from Chicago to Centralia with the distances and the 1959 fares from Chicago to the various stations.

Miles	Stations	Fare from Chicago
0	Chicago	
54.5	Kankakee	\$1.51
79.8	Gilman	2.21
112.4	Rantoul	3.11
126.5	Champaign	3.52
171.0	Mattoon	4.77
197.9	Effingham	5.51
251.1	Centralia	6.97

1. Do these data support the statement that the cost of a trip by rail varies directly as the distance traveled?

2. If the cost (c) of travel from Chicago toward Centralia is proportional to the distance (d) traveled, what is the factor of proportionality? Write a formula for c in terms of d.

3. The fares listed in the table do not include a 10% Federal tax on transportation. Suppose the tax were added to the fares. If the fare without tax varies directly as distance traveled, does the fare including tax vary directly as the distance traveled?

4. Train No. 25 leaves Gilman at 2:31 a.m., arrives at Rantoul at 3:23 a.m., and stays there five minutes; it arrives at Champaign at 3:55 a.m.. Do these data support the statement that the time it takes to travel from one station to the next is proportional to the distance between stations?

<u>G</u>. Suppose the number (H) of hours to do a certain job varies inversely as the number (M) of machines working.

1. (a) Suppose 10 machines can complete a job in 3 hours. How long will it take 8 machines to complete the job?

   (b) Write a formula for H in terms of M.

2. (a) Suppose 8 machines can complete a job in 4 hours. Write a formula for H in terms of M.

   (b) If 2 machines break down after 1 hour of work, how long will it take the other machines to complete the job?

<u>H</u>. 1. The area-measure of a rectangle having a fixed length-measure varies directly as the width-measure. If the area of one such rectangle is 186 square inches and the rectangle is 12 inches wide, how wide is such a rectangle if its area is 170 square inches?

2. Write a formula for the area-measure (A) of such a rectangle in terms of its width-measure (w).

3. Write a formula for w in terms of A.

4. Compute the ratio of A to w. Of w to A.

JOINT VARIATION

Consider the rectangle formula:

$$P = 2(\ell + w)$$

Does P vary directly as ℓ? To answer this question we compute the ratio of P to ℓ.

$$\frac{P}{\ell} = \frac{2(\ell + w)}{\ell} = 2(1 + \frac{w}{\ell}).$$

If $\frac{w}{\ell}$ is not a constant, $\frac{P}{\ell}$ is not a constant. Hence, P does not vary directly as ℓ. Similarly, P does not vary directly as w.

Now, consider two rectangles, r_1 and r_2, for which

$$\ell_2 + w_2 = 5(\ell_1 + w_1).$$

What can you say about P_2 and P_1? Do you see that the variable quantity P varies directly as the sum of the variable quantities ℓ and W? What is the ratio of P to $\ell + w$?

Another case like this occurs in connection with circles. Since $A = \pi r^2$, A does not vary directly as r [the ratio $\frac{A}{r}$ is πr which is not a constant.] But, A does vary directly as the square of r because $\frac{A}{r^2} = \pi$, and π is a constant. Similarly, for rectangles, A varies directly as the product of ℓ and w. [What is the factor of variation?]

Suppose P varies directly as the product of Q and R. This means that there is a constant $k \neq 0$ such that

$$(*) \qquad P = kQR.$$

A case of direct variation like this involving the product of two variable quantities is sometimes called joint variation. One says that P varies jointly as Q and R. Now, if 0 is not a value of Q, it follows from (*) that

$$R = \frac{1}{k} \cdot \frac{P}{Q},$$

or, using 'k′' as a name for the constant $\frac{1}{k}$,

$$(**) \qquad R = k' \cdot \frac{P}{Q}.$$

This tells us that R varies directly as the ratio of P to Q. And, this situation is usually described by saying that

R varies directly as P and inversely as Q.

Example 1. Suppose z varies jointly as x and y. If $z_1 = 4$ when $x_1 = 5$ and $y_1 = 8$, what is z_2 when $x_2 = 6$ and $y_2 = 15$? The expression 'z varies jointly as x and y' can be translated to 'z varies directly as the product of x and y'. So, there is a constant $k \neq 0$ such that

$$z = kxy.$$

We can solve the given problem by first finding the fixed value of k:

$$4 = k \cdot 5 \cdot 8$$

$$k = \frac{1}{10},$$

and then using the formula '$z = \frac{1}{10}xy$':

$$z_2 = \frac{1}{10} \cdot 6 \cdot 15$$

$$z_2 = 9$$

Or, we can set up a proportion:

$$\frac{z_1}{z_2} = \frac{x_1 y_1}{x_2 y_2},$$

and substitute:

$$\frac{4}{z^2} = \frac{5 \cdot 8}{6 \cdot 15},$$

$$z_2 = 4 \cdot \frac{6 \cdot 15}{5 \cdot 8} = 9$$

Example 2. Suppose z varies directly as x and inversely as y. If $z_1 = 3$ when $x_1 = 9$ and $y_1 = 4$, what is z_2 when $x_2 = 7$ and $y_2 = 5$? The expression 'z varies directly as x and inversely as y' can be translated to 'z varies directly as the ratio of x to y'. So, there is a constant $k \neq 0$ such that

$$z = k\frac{x}{y}.$$

We can find the fixed value of k as follows:

$$3 = k\frac{9}{4},$$

$$k = \frac{4}{3}$$

Then, $$z_2 = \frac{4}{3} \cdot \frac{7}{5} = \frac{28}{15}.$$

We can also set up the proportion:

$$\frac{z_1}{z_2} = \frac{\frac{x_1}{y_1}}{\frac{x_2}{y_2}}$$

and derive from it:

$$\frac{z_1}{z_2} = \frac{x_1}{y_1} \cdot \frac{y_2}{x_2}$$

So,

$$\frac{z_1}{z_2} = \frac{x_1}{x_2} \cdot \frac{y_2}{y_1}$$

Substituting and solving, we get:

$$\frac{3}{z_2} = \frac{9}{7} \cdot \frac{5}{4},$$

$$z_2 = 3 \cdot \frac{7}{9} \cdot \frac{4}{5} = \frac{28}{15}$$

EXERCISES

A. For each of the following cases of variation, write a formula which expresses one of the variable quantities in terms of the others. In each case, use 'k' to name the constant whose value is the factor of variation.

Sample. y varies inversely as the square of x.

Solution. By the definition of inverse variation,

$$yx^2 = k.$$

So, the required formula is:

$$y = \frac{k}{x^2}$$

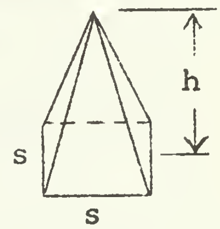
1. z varies jointly as x and the square of y.
2. z varies directly as the cube of x and inversely as y.
3. z varies jointly as x and y and inversely as w.
4. The volume-measure (V) of a circular cone varies jointly as the square of the radius (r) of its base and its height-measure (h). [What is the value of k?]

5. The volume-measure (V) of a sphere is directly proportional to the cube of its radius (r)

6. The force (F) of gravitational attraction between two bodies of masses m_1 and m_2 varies jointly as their masses and inversely as the square of the distance (d) between them.

7. The number (I) of amperes of current in an electric circuit varies directly as the number (E) of volts in the electromotive force and inversely as the number (R) of ohms of resistance.

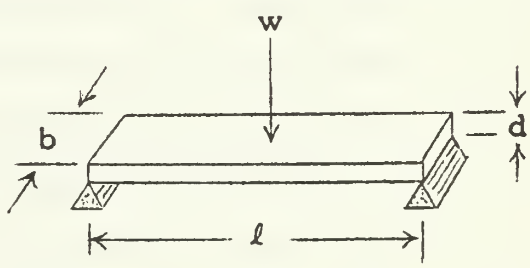
8. The volume-measure (V) of a square pyramid varies directly as the height-measure (h) and the square of the side-measure (s) of the base.



9. (a) The velocity (v) of a body falling from rest varies directly as the number (t) of seconds of fall.

   (b) The distance (s) a body falls from rest varies directly as the square of the number (t) of seconds of fall.

10. (a) The safe load (w) for a horizontal rectangular beam supported at the ends varies jointly as the breadth (b) and the square of the depth (d), and inversely as the distance (ℓ) between supports.



   (b) The deflection (D) of the middle of the beam varies jointly as the load (w) and the cube of the distance (ℓ) between supports, and inversely as the product of the breadth (b) and the cube of the depth (d).

B. 1. w varies directly as u and inversely as v. The value of w is 7 when the value of u is 10 and the value of v is 5.

   (a) What is the value of w when the value of u is 8 and the value of v is 2?

   (b) What is the value of u when the value of w is 28 and the value of v is 10?

2. A varies directly as B and inversely as the square of C.

   (a) If $A_1 = 5$ and $B_1 = 8$ and $C_1 = 4$, what is A_2 if $B_2 = 12$ and $C_2 = 10$?

   (b) If $A_1 = 3$ when $B_1 = 9$ and $C_1 = \sqrt{3}$, what is B_2 if $A_2 = 4$ and $C_2 = 5$?

3. M is proportional to the square of N and inversely proportional to the cube of P.

   (a) If the value of M is 1 when the value of N is 4 and the value of P is 2, what is the factor of variation?

   (b) Suppose N varies directly as the square of R, and P varies directly as R. Write a formula for M in terms of R.

   (c) Suppose N varies directly as the square root of S, and P varies inversely as the cube root of S. Write a formula for M in terms of S.

4. The force (F) of wind blowing on a flat surface [such as a billboard or a sail] and at right angles to it, is jointly proportional to the area (A) of the surface and the square of the speed (r) of the wind. The pressure of the wind is 5 pounds per square foot when the wind is blowing at 20 miles per hour.

   (a) What is the total force of a 30-mile per hour wind on a billboard 10 feet by 30 feet?

   (b) What is the total force of a 30-mile per hour wind on a bass drum 3.5 feet in diameter?

☆ 5. Kepler's third law of planetary motion states that the time (t) of one revolution of a planet around the sun is proportional to the square root of the cube of the distance (d) between the planet and the sun. If the distance between Mars and the sun is about 1.5237 times the distance between Earth and the sun, how many Earth days are there in a Martian year? [An Earth year is 365.26 Earth days.]

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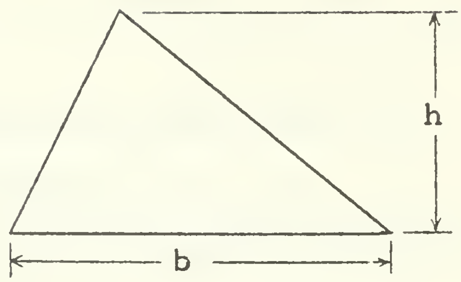


The area-measure of a triangle varies jointly as its height-measure and its base-measure. If k is a constant whose value is the factor of variation then a formula which expresses A in terms of h and b is:

$$A = khb$$

[Do you recall the value of k from an earlier course?]

Now, consider a triangle for which the values of A, h, and b are A_1, h_1, and b_1, and a second triangle for which the values are A_2, h_2, and b_2. If $h_2 = 2h_1$ and $b_2 = 3b_1$, what can you say about A_2 and A_1? [In other words, if you double the height and triple the base of a triangle, what happens to the area?] Since $A_2 = kh_2b_2$, it follows that $A_2 = k(2h_1)(3b_1) = 6(kh_1b_1)$. Since $A_1 = kh_1b_1$, we see that $A_2 = 6A_1$. [So, the area is multiplied by 6.]

What happens to the area of a triangle if you increase the height by 50% and decrease the base by 25%? Here, $h_2 = h_1 + 0.5h_1 = 1.5h_1$, and $b_2 = b_1 - 0.25b_1 = 0.75b_1$. So,

$$\begin{aligned} A_2 = kh_2b_2 &= k(1.5h_1)(0.75b_1) \\ &= 1.125(kh_1b_1) \\ &= 1.125A_1 \\ &= A_1 + 0.125A_1. \end{aligned}$$

Thus, the area increases by 12.5% [or, by $\frac{1}{8}$].

* * *

C. The perimeter (P) of a square varies directly as its side-measure (s).

1. What change takes place in the perimeter if you triple the side?

2. What change takes place in the perimeter if you decrease the side by 50%?

3. What change takes place in the side if you increase the perimeter by 80%?

☆ 4. What change takes place in the perimeter if you increase the side by x%?

D. The area-measure of a square varies directly as the square of its side-measure.

1. What change takes place in the area if you double the side?

2. What happens to the area if you increase the side by 40%?

3. What happens to the area if you decrease the side by 30%?

☆ 4. What change takes place in the area if you increase the side by x%?

E. The area-measure of a rectangle varies jointly as the length-measure and the width-measure.

1. What happens to the area if you double each dimension?

2. What happens to the area if you increase the length by 50% and decrease the width by 50%?

3. What happens to the area if you increase the length by a third and decrease the width by a fourth?

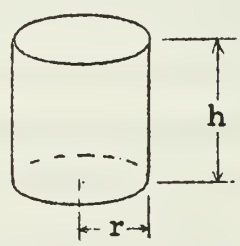
4. What happens to the area if you increase the length by 100% and decrease the width by 50%?

5. What change takes place in the width if you double the area and increase the length by 25%?

6. What change takes place in one of the dimensions if the other is increased by 60% and the area remains the same?

☆ 7. If you increase one dimension by x% and the other by y%, what change takes place in the area?

F. The volume-measure (V) of a circular cylinder varies jointly as its height-measure (h) and the square of the radius (r) of its base.



1. Write a formula for V in terms of h and r and using 'k' as the name of the constant whose value is the factor of variation.

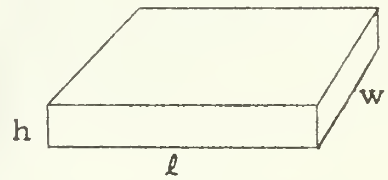
2. If you double the height but do not change the radius, what change takes place in the volume?

3. If you double the radius but do not change the height, what change takes place in the volume?

4. If you increase the radius by 50% and decrease the volume by 50%, what change takes place in the height?

☆5. If you increase the radius by x% and leave the height unchanged, what happens to the volume?

G. The length-measure (ℓ) of a rectangular block varies directly as the volume-measure (V) and inversely as the product of the width-measure (w) and the height-measure (h).



1. If $V_2 = 3V_1$, $w_2 = 3w_1$, and $h_2 = 3h_1$, what can you say about ℓ_2 and ℓ_1?

2. If you double the volume, the width, and the height, what change takes place in the length?

3. If you increase each of the dimensions by 50%, what change takes place in the volume?

4. If you increase both the length and the width by 50%, and decrease the height by 50%, what change takes place in the volume?

5. If you decrease the height by 25% and double the width, what change must you make in the length to keep the volume unchanged?

☆6. If you increase the length by p%, the width by q%, and the height by r%, what change takes place in the volume?

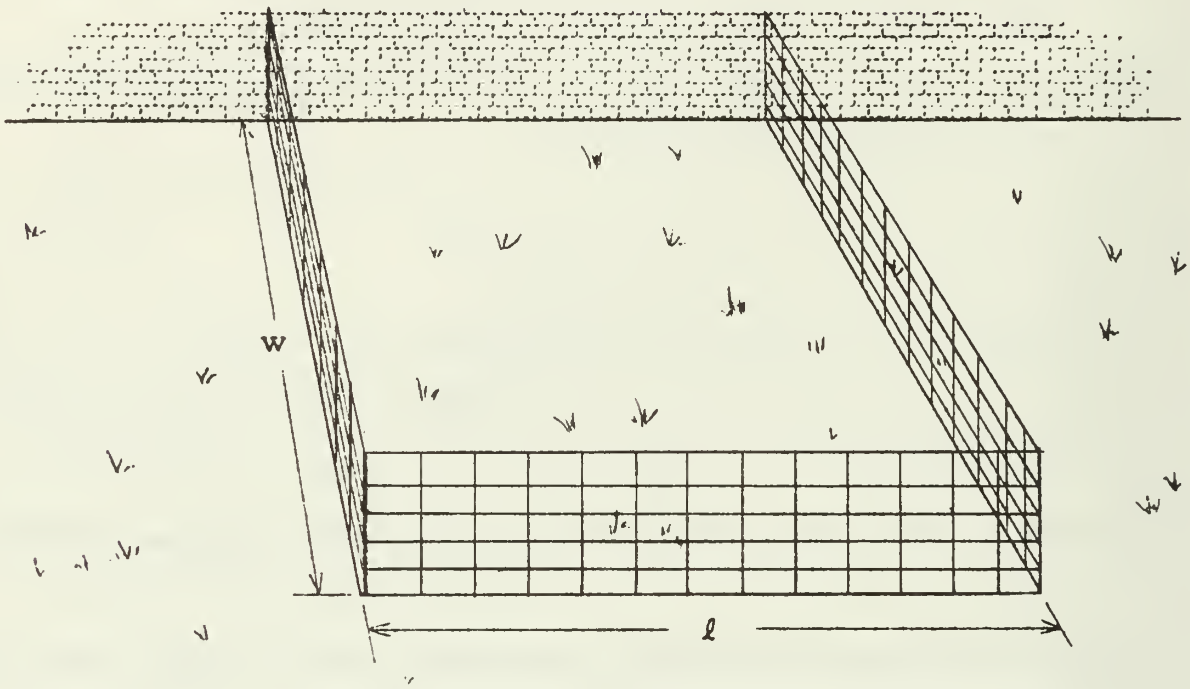
H. Suppose A varies jointly as B and C and inversely as the square of D.

1. If you double D and leave B and C unchanged what happens to A?

2. If you triple each of B, C, and D, what change takes place in A?

[Supplementary exercises are in Part T, pages 5-269 through 5-271.]

5.09 Quadratic functions.--A farmer wants to fence off a rectangular pen using part of an existing brick wall for one of the sides. He has 120 feet of fencing which he can use for this purpose.



What should the dimensions be if the fence is to enclose as large an area as possible?

Here is a table listing some of the possible dimensions and the corresponding area-measures.

w	ℓ	A
1	118	118
2	116	232
3	114	342
10	100	1000
20	80	1600
50	20	1000
58	4	232

It seems reasonable that there are values of ℓ and w which correspond with a value of A which is larger than any other value of A. Our problem is to find these values of ℓ and w.

We know that

(1) $$A = \ell w.$$

We also know that the 120 feet of fencing must be used for one length

and two widths of the rectangular pen. So, for the set of all rectangular pens of this design,

$$(2) \qquad \ell + 2w = 120.$$

We transform (2) to get:

$$(3) \qquad \ell = 120 - 2w$$

So, (1) tells us that A is a function of (ℓ, w), and (3) tells us that ℓ is a function of w. Given a value of w, the value of ℓ is determined, and so is the value of A. Hence, A is a function of w. We can find out what function A is of w by substituting (3) into (1):

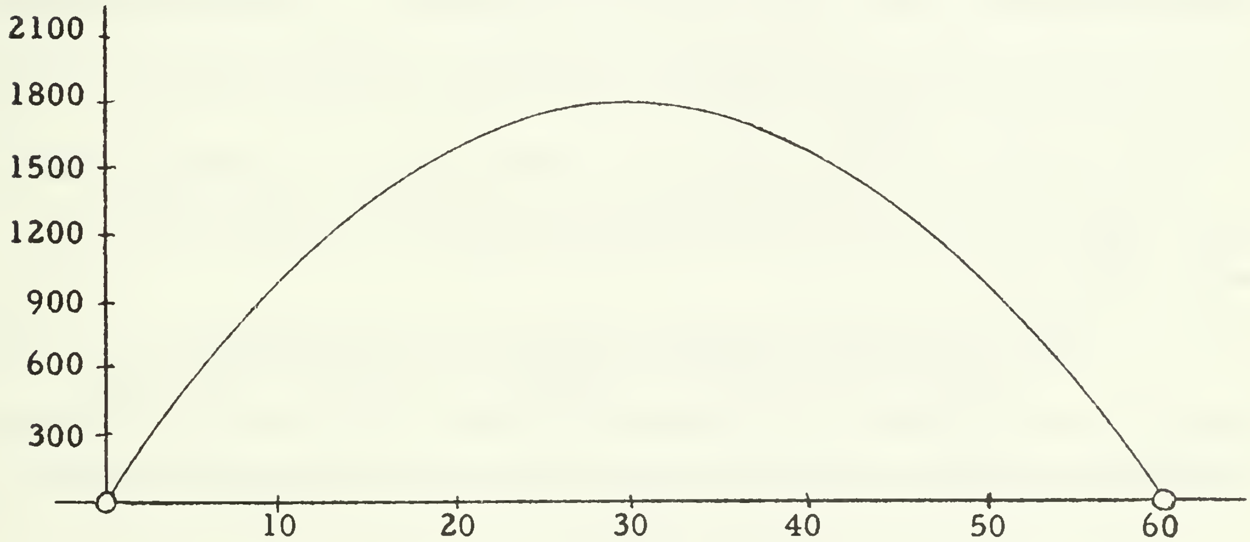
$$(4) \qquad A = (120 - 2w)w$$

Thus, the ordered pairs of corresponding values of w and A are the ordered pairs in

$$\{(x, y),\ 0 < x < 60:\ y = (120 - 2x)x\}.$$

Call this function 'f'. Now, our job is to find that argument of f for which f has its largest value. This argument will give us the required width, and then we can use (3) to find the required length.

Let's make a graph of f.



An examination of the graph suggests that the maximum value of f corresponds with the argument 30. So, it seems that the pen of maximum area is the one which is 30 feet wide and 60 feet long. [What is this maximum area?]

In this section you will learn a quicker and surer way to solve a problem like this.

QUADRATIC FUNCTIONS OF ONE REAL VARIABLE

Up to now we have considered two special classes of functions--the constant functions and the linear functions.

f is a <u>constant</u> <u>function</u> if and only if
there is a number a such that $f = \{(x, y): y = a\}$.

f is a <u>linear</u> <u>function</u> if and only if
there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$.

We shall now consider another class of functions which are called <u>quadratic functions of one real variable</u>, or, for short, <u>quadratic functions</u>.

f is a <u>quadratic</u> <u>function</u>
if and only if
there are numbers $a \neq 0$, b, and c such that
$f = \{(x, y): y = ax^2 + bx + c\}$.

You have already seen some examples of quadratic functions. For example, the area-measure of a square is a quadratic function of its side-measure. The quadratic function in question is $\{(x, y): y = x^2\}$, which we have called the <u>squaring</u> <u>function</u>. In the fencing problem discussed earlier, the area-measure of the pen is a quadratic function of the width-measure, the quadratic function in question being $\{(x, y): y = (120 - 2x)x\}$.

How can you tell that $\{(x, y): y = (120 - 2x)x\}$ is a quadratic function? Just use the definition. If you can produce numbers $a \neq 0$, b, and c such that

$$\{(x, y): y = (120 - 2x)x\} = \{(x, y): y = ax^2 + bx + c\},$$

then the given function is a quadratic function. As in the case of linear functions, you obtain these numbers by transforming the set selector.

$$y = (120 - 2x)x$$
$$y = 120x - 2x^2$$
$$y = -2x^2 + 120x + 0$$

Hence, $a = -2 \neq 0$, $b = 120$, and $c = 0$. So, $\{(x, y): y = (120 - 2x)x\}$ is a quadratic function.

EXERCISES

A. For each set listed, tell whether it is a constant function, a linear function, a quadratic function, or none of these.

1. $\{(x, y): y = 3(2 - x)\}$
2. $\{(x, y): y = 6 - x^2\}$
3. $\{(x, y): y + 2x^2 = 5 - 3x\}$
4. $\{(x, y): 3(x - 5) = x(x - 1)\}$
5. $\{(x, y): y = (x - 2)^2\}$
6. $\{(x, y), x > 0: y = 3x^2 + 2x - 7\}$
7. $\{(x, y): y = -(x + 3)^2 - 5\}$
8. $\{(x, y): y - 3x = (x - 3)^2\}$
9. $\{(x, y): y = (x + 2)(x - 3) - (x - 5)(x + 4)\}$

B. Consider the functions f, g, and h where

$$f = \{(x, y): y = x^2\},\ g = \{(x, y): y = x\},\ \text{and } h = \{(x, y): y = 1\}.$$

1. Graph f, g, and h on the same chart.
2. Use the graphs you drew in Exercise 1 to help you make quick sketches of the graphs of the functions listed below.

(a) $f + h$ (b) $f + 2h$ (c) $f - 5h$

(d) $f + g$ (e) $f + 2g$ (f) $f - 6g$

(g) $-f$ (h) $-(f + 2h)$ (i) $-(f + 4g)$

C. Sketch the graphs of the functions defined by these equations.

1. $y = x^2$
2. $y = (x - 3)^2$
3. $y = (x + 1)^2$
4. $y = x^2 + 1$
5. $y = (x - 3)^2 + 1$
6. $y = (x + 1)^2 - 5$

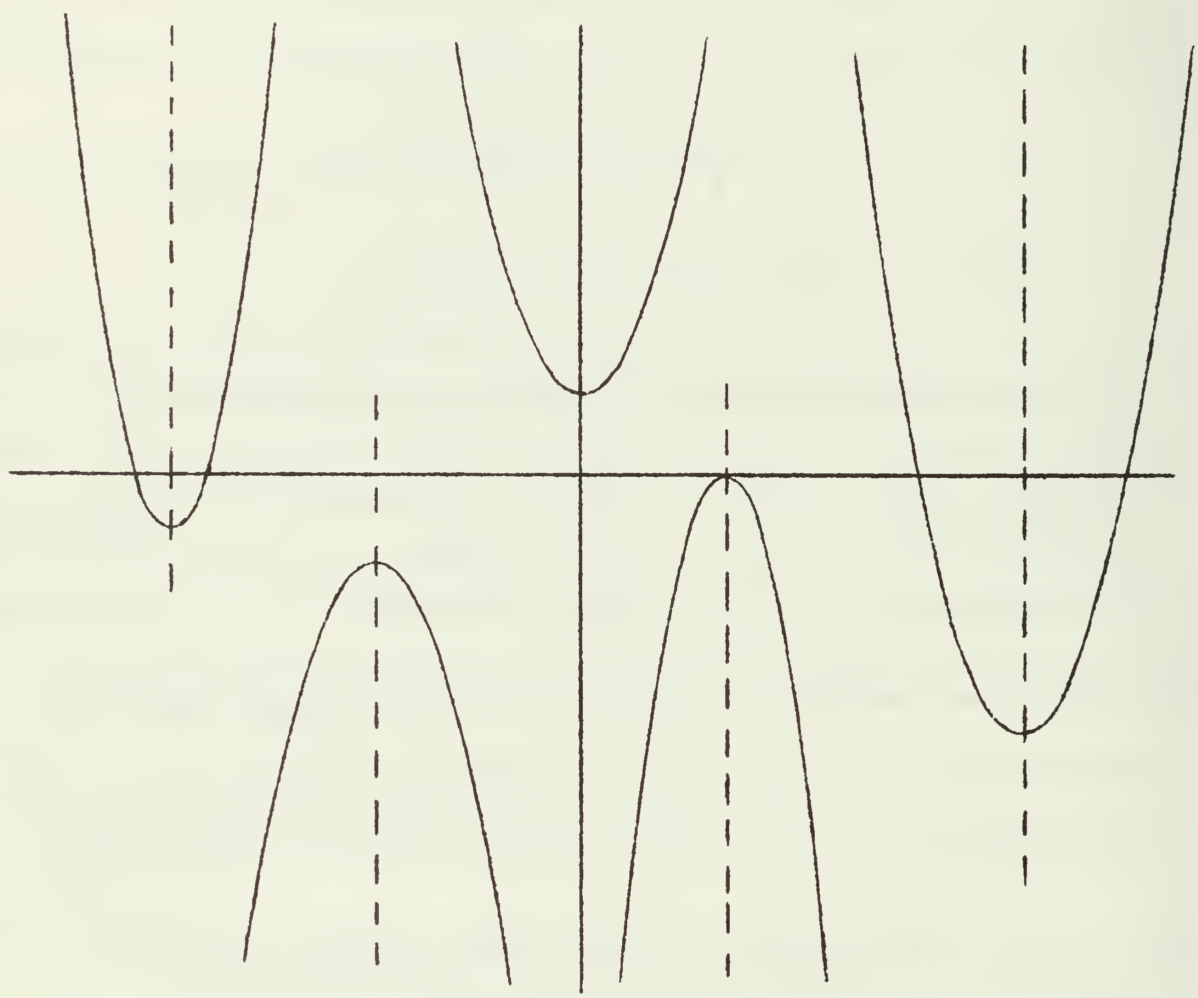
D. Find the argument which corresponds with the minimum value for each of the quadratic functions you graphed in Part C.

E. Find the argument which corresponds with the extreme value [maximum or minimum] of each function listed below. Then, graph the function.

1. $\{(x, y): y = (x - 3)^2 + 5\}$
2. $\{(x, y): y = (x - 5)^2 - 9\}$
3. $\{(x, y): y = -x^2\}$
4. $\{(x, y): y = -(x - 3)^2\}$
5. $\{(x, y): y = -(x + 7)^2 + 4\}$
6. $\{(x, y): y = x^2 - 2x + 1\}$

GRAPHING A QUADRATIC FUNCTION

In the preceding exercises you practiced graphing quadratic functions. You may have noted that all graphs of quadratic functions have the same shape. In fact, they are all examples of curves called parabolas. A parabola which is the graph of a quadratic function is symmetric with respect to some vertical line. Here are examples of graphs of quadratic functions. Note the graph of the axis of symmetry in each case. Also, notice that each quadratic function has an "extreme point" which is the point in the intersection of the quadratic function and



its axis of symmetry. The graph of the extreme point is the vertex of the parabola.

Just as in graphing linear functions it was helpful to use the notions of slope and intercept, so in graphing quadratic functions it would be helpful to know the function's axis of symmetry, its extreme point, and whether the parabola opens upward or downward. It would also be helpful to know the intersection of the function with the y-axis and the

intersection with the x-axis. Since a quadratic function is determined by the values of 'a', 'b', and 'c' in the defining equation:

$$y = ax^2 + bx + c,$$

we should expect to be able to use these numbers to give all of the information mentioned above. [One item of information which is available immediately is the intersection of the function and the y-axis. How many points are there in this intersection? What are the components?]

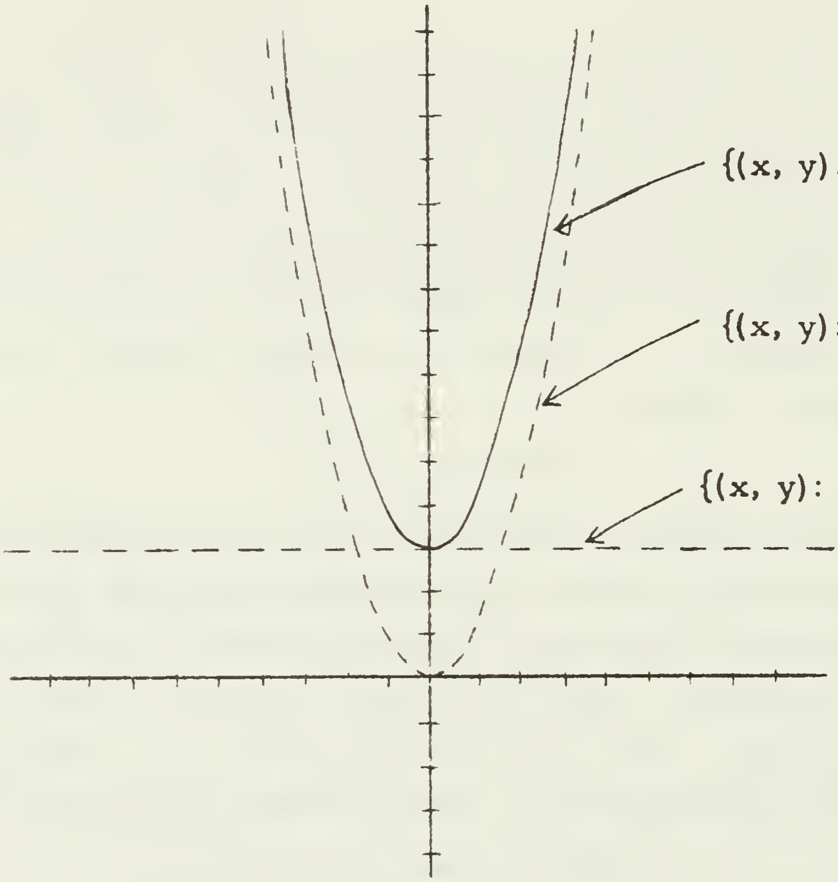
Let's explore a bit to find an easy way of discovering the axis of symmetry. [Why is it helpful to know the axis of symmetry in graphing a quadratic function?] Let's take the simplest quadratic function, the squaring function, which is

$$\{(x, y): y = x^2\}.$$

Think of all the horizontal lines which cross the graph of the squaring function. The lowest of these lines contains the graph of (0, 0). Hence, (0, 0) is the extreme point of the squaring function, and its graph is the vertex of the parabola. Each of the other horizontal lines crosses the graph in two places. Pick some horizontal line, say, the one which is 9 units above the horizontal axis. What are the components of the points in which $\{(x, y): y = 9\}$ intersects the squaring function? How long is the interval whose end points are these points? What is the midpoint of this interval? What is the set of all such midpoints? This set is contained in the axis of symmetry of the squaring function. If you hold the edge of a mirror along the graph of the y-axis with the mirror facing the graph of the first quadrant, you can see the image of the graph of the first quadrant portion of the squaring function. And, this image is exactly the graph of the second quadrant portion of the function. An advantage in knowing the axis of symmetry of a quadratic function is that in graphing the function you need plot points on only one side of the graph of the axis of symmetry. For each point so plotted, you can get a second one by reflecting it in the axis of symmetry.

Another point of interest concerning the squaring function is that the graph rises slowly to the right and to the left from the graph of (0, 0). The rate of rise increases as soon as you move away from the vertex and the curve gets steeper and steeper as you continue to move. Can you explain this in terms of the properties of squaring?

Now, consider the quadratic function $\{(x, y): y = x^2 + 3\}$. This is the sum of the squaring function and the constant function $\{(x, y): y = 3\}$. The graph of the sum of these functions has precisely the same shape as



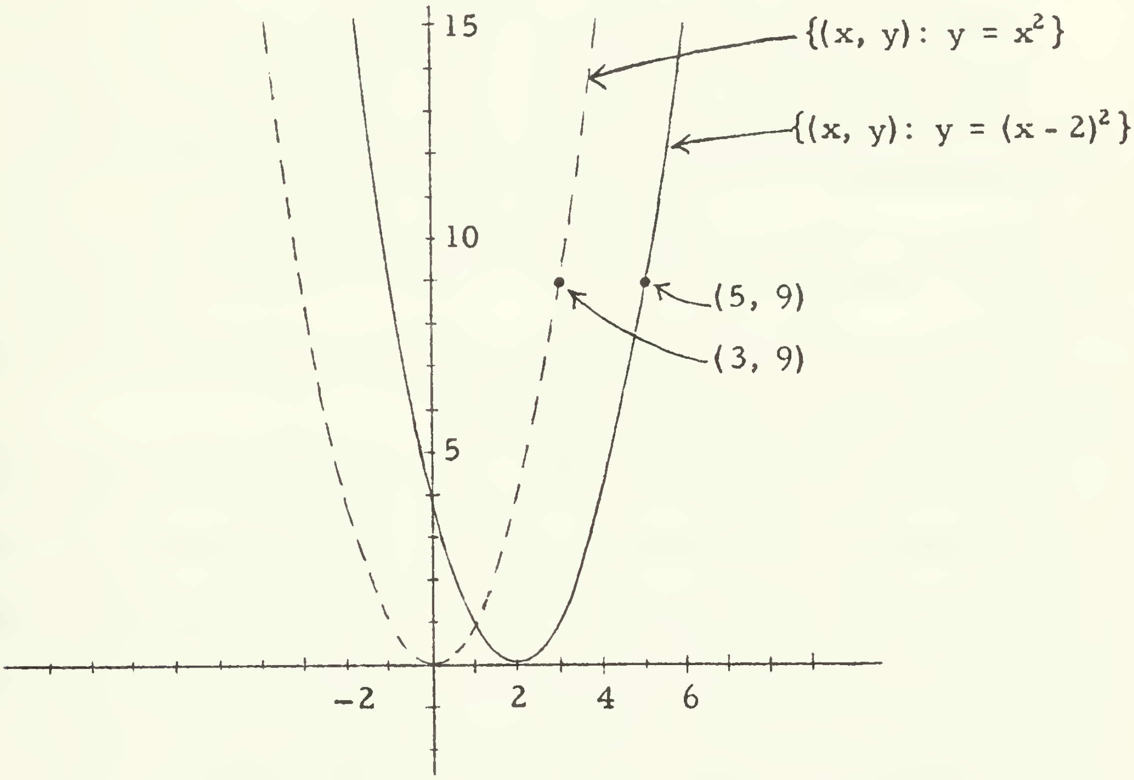
the graph of the squaring function. The only difference is the position of the vertex. The vertex is the graph of (0, 3) instead of the graph of (0, 0). The axis of symmetry is still $\{(x, y): x = 0\}$.

What is the axis of symmetry of $\{(x, y): y = x^2 - 5\}$? What is the extreme point of this function?

Let's consider the opposite of the squaring function, $\{(x, y): y = -x^2\}$. What is its axis of symmetry? What is its extreme point? Answer these questions for the quadratic functions $\{(x, y): y = -x^2 + 5\}$ and $\{(x, y): y = -x^2 - 4\}$. Answer them for $\{(x, y): y = -(x^2 + 4)\}$.

In general, for each q, the axis of symmetry of the quadratic function $\{(x, y): y = x^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is __________, and its graph opens ______________; the axis of symmetry of $\{(x, y): y = -x^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is ____________, and its graph opens _______________.

Now, let's consider the quadratic function $\{(x, y): y = (x - 2)^2\}$. This function may remind us of the squaring function. In fact we see that (5, 9) belongs to <u>this</u> function just because (5 - 2, 9) belongs to the squaring function. Turning this around: Because (3, 9) belongs to the squaring function, we know that $(3 + 2, 9) \in \{(x, y): y = (x - 2)^2\}$. So, we can obtain a graph of $\{(x, y): y = (x - 2)^2\}$ just by shifting a graph of the squaring function 2 units to the right.



[Another way to get a graph of $\{(x, y): y = (x - 2)^2\}$ is to draw a graph of the squaring function and then re-draw the graph of the vertical axis 2 units to the <u>left</u>.] What is the axis of symmetry of $\{(x, y): y = (x - 2)^2\}$? What is its extreme point? Answer these questions for the quadratic functions $\{(x, y): y = -(x - 2)^2\}$, $\{(x, y): y = (x + 3)^2\}$, $\{(x, y): y = -(x + 3)^2\}$, and $\{(x, y): y = (x - 2)^2 + 5\}$.

In general, for each p, the axis of symmetry of the quadratic function $\{(x, y): y = (x - p)^2\}$ is $\{(x, y):$ ____________$\}$, the extreme point is _________, and its graph opens _________; the axis of symmetry of $\{(x, y): y = -(x - p)^2\}$ is $\{(x, y):$ ____________$\}$, the extreme point is _________, and its graph opens _________.

For each p and q, the axis of symmetry of the quadratic function $\{(x, y): y = (x - p)^2 + q\}$ is $\{(x, y):$ __________$\}$, the extreme point is ______, and its graph opens ___________; the axis of symmetry of $\{(x, y): y = -(x - p)^2 + q)\}$ is $\{(x, y):$ ___________$\}$, the extreme point is _____, and its graph opens ___________.

Explain how to get a graph of $\{(x, y): y = (x - 3)^2 + 5\}$ by shifting a graph of the squaring function.

Consider, now, the quadratic function

$$\{(x, y): y = (x - 3)^2 - 3^2\}$$

or, more simply, $\{(x, y): y = x^2 - 6x\}$. What is its axis of symmetry? Its extreme point? For each p, the axis of symmetry of

$$\{(x, y): y = (x - p)^2 - p^2\},$$

or, more simply, $\{(x, y): y = x^2 - 2px\}$ is $\{(x, y):$ ____________$\}$.

What are the axis of symmetry and the extreme point of each of the quadratic functions listed below?

(a) $\{(x, y): y = (x - 2)^2 - 2^2\}$ (b) $\{(x, y): y = (x + 2)^2 - 2^2\}$

(c) $\{(x, y): y = x^2 - 4x\}$ (d) $\{(x, y): y = x^2 + 4x\}$

(e) $\{(x, y): y = x^2 - 5x\}$ (f) $\{(x, y): y = x^2 + 5x\}$

(g) $\{(x, y): y = x^2 - 5x + 3\}$ (h) $\{(x, y): y = x^2 + 5x - 7\}$

(i) $\{(x, y): y = -(x^2 - 3x)\}$ (j) $\{(x, y): y = -(x^2 + 3x)\}$

(k) $\{(x, y): y = -x^2 + 3x + 4\}$ (ℓ) $\{(x, y): y = -x^2 - 3x - 5\}$

For each b, the axis of symmetry of $\{(x, y): y = x^2 + bx\}$ is $\{(x, y):$ ___________$\}$. For each b and c, the axis of symmetry of $\{(x, y): y = x^2 + bx + c\}$ is $\{(x, y):$ ___________$\}$, the extreme point is ______, and its graph opens ___________.

For each b and c one can obtain a graph of $\{(x, y): y = x^2 + bx + c\}$ by shifting a graph of the squaring function until the graph of its vertex is on the graph of (___, ______).

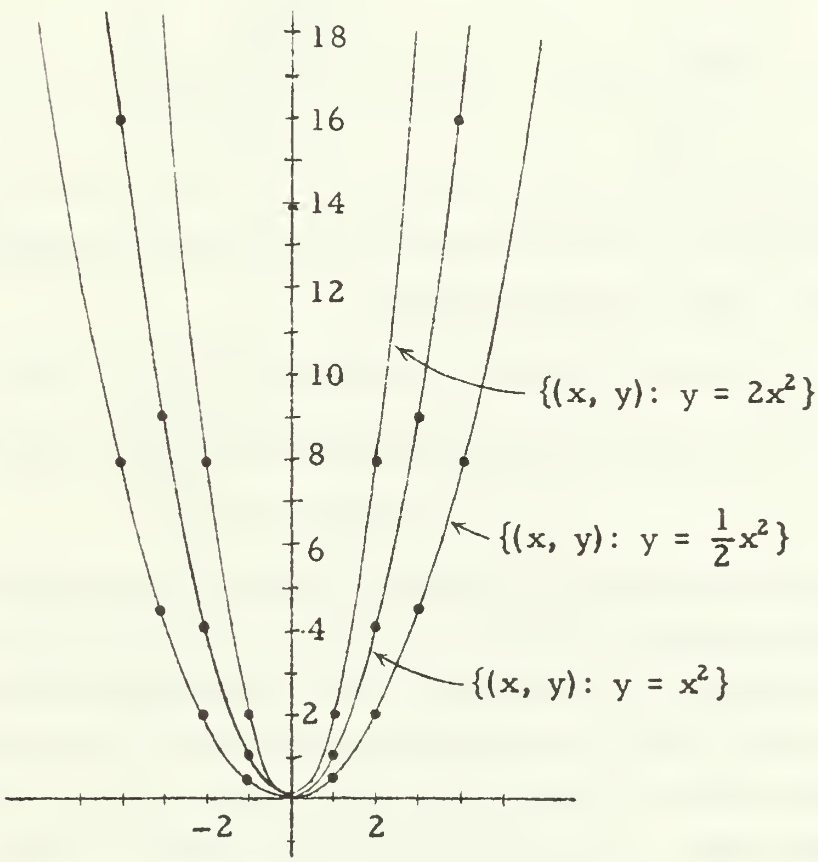
Up to now we have been discussing quadratic functions whose defining equations are of the form:

$y = x^2 + bx + c$ or: $y = -x^2 + bx + c$

You know how to use the numbers b and c to find the axis of symmetry

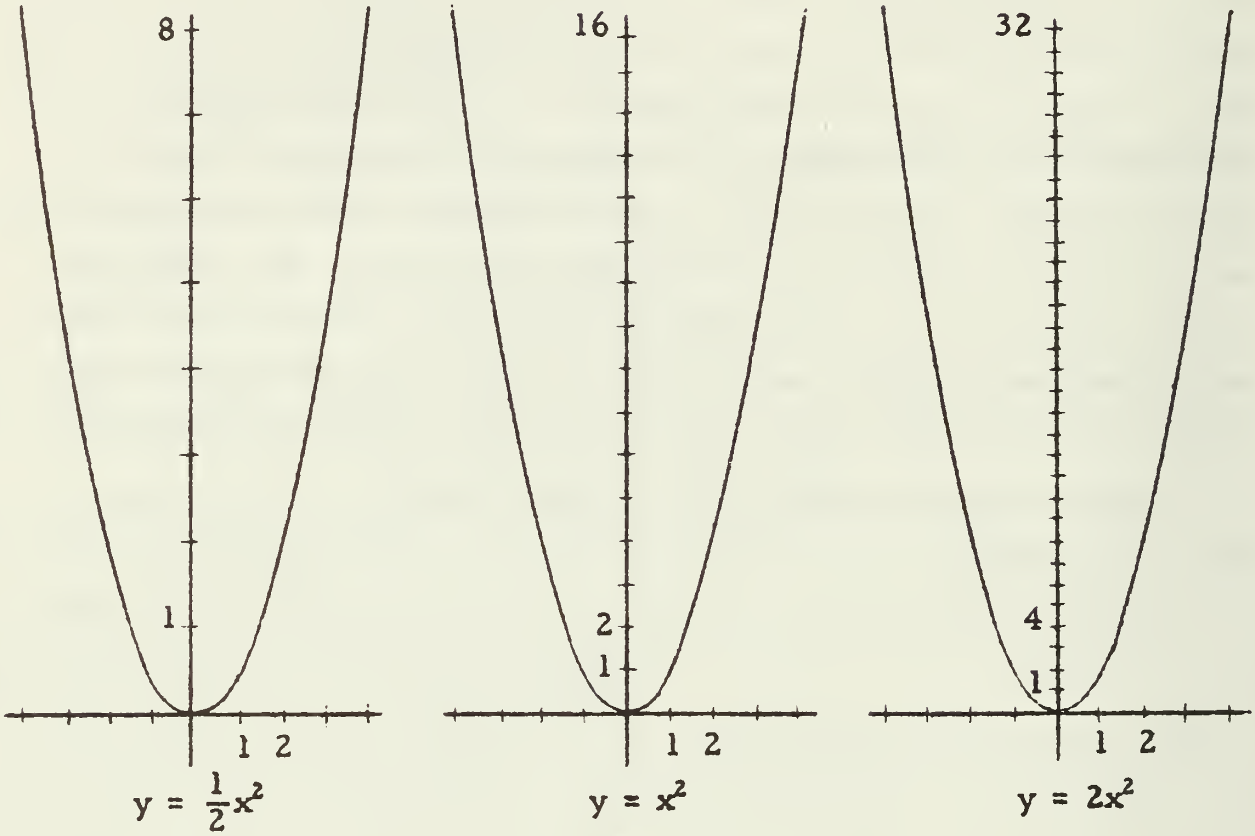
and the extreme point of any quadratic function whose defining equation is of either of these forms. You also know that a graph of any such quadratic function can be obtained by merely shifting a graph of the squaring function [and, in case the defining equation is of the second form, tipping the graph upside down]. As a practical application of this last piece of knowledge, you can make a "parabolic ruler": Draw carefully a graph of the squaring function, paste your graph paper on heavy cardboard, and cut along the graph. Now you have a ruler which you can use to draw graphs of any quadratic functions whose defining equations are of either of the forms given near the bottom of page 5-174. But, how about other quadratic functions?

Here are graphs of $\{(x, y):\ y = \frac{1}{2}x^2\}$, $\{(x, y):\ y = x^2\}$, and $\{(x, y):\ y = 2x^2\}$:



It is easy to see how, for example, to obtain a graph of $\{(x, y):\ y = \frac{1}{2}x^2\}$ once one has a graph of the squaring function. Just move each point of the graph of the squaring function half-way to the graph of the x-axis. Instead of moving the points down, you could move

the scale markings on the graph of the y-axis up. Here are the graphs of the same three functions:



You can now use your parabolic ruler to draw a graph of any quadratic function. Let's try an example.

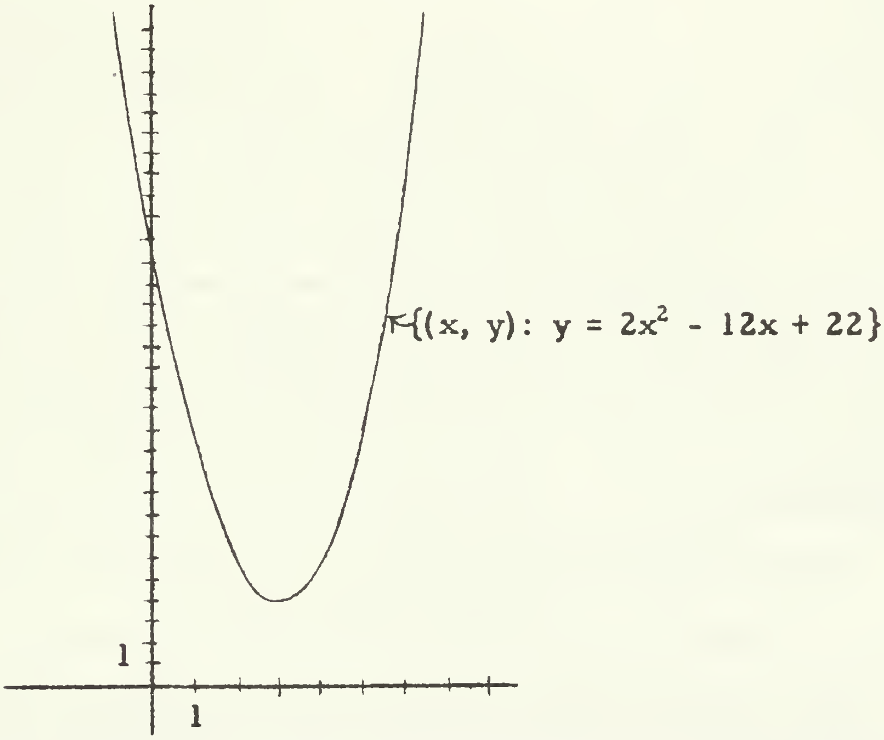
Graph the quadratic function $\{(x, y): y = 2x^2 - 12x + 22\}$.

First, we transform the defining equation to an equivalent one:

$$y = 2(x^2 - 6x + 11)$$

This shows us that we could get a graph of the given function by first drawing a graph of $\{(x, y): y = x^2 - 6x + 11\}$ and then changing the scale markings on the graph of the y-axis so that the unit length is just half as long as the unit length on the graph of the x-axis. If you used the same scale on the graph of the x-axis that you used in making your parabolic ruler, then you could use the ruler to draw a graph of the given function. Since the axis of symmetry of $\{(x, y): y = x^2 - 6x + 11\}$ is $\{(x, y): x = 3\}$ and the extreme point is (3, 2), the axis of symmetry of $\{(x, y): y = 2(x^2 - 6x + 11)\}$ is $\{(x, y): x = 3\}$ and <u>its</u> extreme point is (3, 4). [Explain]. Also, both graphs open upward. So, you can

draw a graph of $\{(x, y): y = 2x^2 - 12x + 22\}$ by choosing proper scales on the graphs of the axes and placing your parabolic ruler with its vertex on the graph of (3, 4) and its center line on the upper part of the graph of $\{(x, y): x = 3\}$.



How would you place your ruler to draw graphs of the quadratic functions listed below?

(a) $\{(x, y): y = 2x^2 - 8x + 7\}$

(b) $\{(x, y): y = -2x^2 + 8x - 1\}$

(c) $\{(x, y): y = 2x^2 + 6x + 4\}$

(d) $\{(x, y): y = -2x^2 - 6x - \frac{1}{2}\}$

(e) $\{(x, y): y = 2x^2 + 12x + 7\}$

(f) $\{(x, y): y = -2x^2 - 12x - 11\}$

(g) $\{(x, y): y = 2x^2 + 8x + 9\}$

(h) $\{(x, y): y = -2x^2 - 8x + 1\}$

For all $a \neq 0$, b, and c, the axis of symmetry of

$$\{(x, y): y = ax^2 + bx + c\}$$

is ________________; the extreme point is ________________; the graph opens upward if _____, and opens downward if _____.

COMPLETING THE SQUARE

In graphing the quadratic function $\{(x, y): y = 2x^2 - 12x + 22\}$ we found that its axis of symmetry is $\{(x, y): x = 3\}$, that its extreme point is $(3, 4)$, that its graph opens upward, and that, to use the parabolic ruler to draw a graph of this function, we had to use a unit length on the graph of the y-axis half as long as the unit length on the graph of the x-axis. Now, consider the quadratic function $\{(x, y): y = 2(x - 3)^2 + 4\}$. What is its axis of symmetry? Its extreme point? Does its graph open upward or downward? What unit length would you choose on the graph of the y-axis to enable you to use your parabolic ruler in drawing the graph of this function? Are $\{(x, y): y = 2(x - 3)^2 + 4\}$ and $\{(x, y): y = 2x^2 - 12x + 22\}$ different functions?

If the extreme point of $\{(x, y): y = ax^2 + bx + c\}$ is (p, q) then

$$\{(x, y): y = ax^2 + bx + c\} = \{(x, y): y = a(\_\_\_\_\_\_)^2 + \_\_\_\_\}.$$

You now see that each quadratic function has a defining equation of the form:

$$y = a(x - p)^2 + q,$$

and, of course, each such equation for which $a \neq 0$ defines a quadratic function. When you have a defining equation of this form for a quadratic function, it is very easy to describe the important characteristics of the function. Its axis of symmetry is ________________; its extreme point is ________; the function has a minimum value, q, if ________; or a maximum value, _____, if ________; if, in drawing a graph, you use the <u>same</u> unit length on the graphs of the x- and y-axes, then the graph is narrower than the graph of the squaring function if $|a|$ _____, and is broader than the graph of the squaring function if _________.

Evidently, it would be useful to have an easy way of transforming an expression of the form:

$$ax^2 + bx + c$$

to an equivalent expression of the form:

$$a(x - p)^2 + q$$

Let us try to transform the expression:

$$x^2 - 12x + 40$$

to one of the form:

$$(x - p)^2 + q$$

We know that, for each p and q,

$$(x - p)^2 + q$$
$$= x^2 - 2px + p^2 + q.$$

Now, compare this last expression with the given one:

$$\begin{array}{cccccc} x^2 & - & 2px & + & p^2 + q \\ & \updownarrow & \updownarrow & \updownarrow & \\ x^2 & - & 12x & + & 40 \end{array}$$

If you substitute '6' for 'p' in the top expression, it becomes one which is equivalent to:

$$x^2 - 12x + 36 + q$$

Then, substitute '4' for 'q', simplify, and you get the given expression. So, '$(x - 6)^2 + 4$' is equivalent to '$x^2 - 12x + 40$'. [Check this by expanding in '$(x - 6)^2 + 4$' and simplifying.] We can see by inspection that the axis of symmetry of $\{(x, y): y = (x - 6)^2 + 4\}$ is $\{(x, y): x = 6\}$, and that the extreme point is $(6, 4)$.

The process of transforming <u>a quadratic expression</u> in 'x', that is, an expression of the form:

$$ax^2 + bx + c \; [a \neq 0]$$

to one of the form:

$$a(x - p)^2 + q$$

is called <u>completing the square</u>. It is important that you become skillful at completing the square. Not only is this skill useful when you need to graph quadratic functions and [as you will see later] in solving quadratic equations, but it is essential for solving many problems which involve quadratic functions.

On the next page we give some quadratic expressions, and ones equivalent to them obtained by completing the square. You should study them with care.

$x^2 + 6x + 12 \ldots\ldots (x + 3)^2 + 3$

$x^2 - 8x - 5 \ldots\ldots (x - 4)^2 - 21$

$x^2 + 3x + 2 \ldots\ldots (x + \frac{3}{2})^2 - \frac{1}{4}$

$x^2 - x + 7 \ldots\ldots (x - \frac{1}{2})^2 + \frac{27}{4}$

$3x^2 + 12x + 5 \ldots\ldots 3(x + 2)^2 - 7$

$-x^2 + 5x + 4 \ldots\ldots -(x - \frac{5}{2})^2 + \frac{41}{4}$

$15x^2 + 7x - 2 \ldots\ldots 15(x + \frac{7}{30})^2 - \frac{169}{60}$

EXERCISES

A. Recall from Unit 3 the work you did in expanding expressions like '$(x - 7)^2$' and '$(y + 3)^2$'. As in all such manipulations, you gain skill by looking for a short cut and then practicing the short cut until you can use it almost without thinking. Take the expression '$(y + 3)^2$'. Let's expand it the long way:

$$\begin{aligned}
& (y + 3)^2 \\
&= (y + 3)(y + 3) && \text{[Why?]} \\
&= y(y + 3) + 3(y + 3) && \text{[Why?]} \\
&= y^2 + y3 + (3y + 9) && \text{[Why?]} \\
&= y^2 + (3y + 3y) + 9 && \text{[Why?]} \\
&= y^2 + (3 + 3)y + 9 && \text{[Why?]} \\
&= y^2 + 6y + 9
\end{aligned}$$

Now, expand '$(y + 5)^2$', but do it all in one step. Similarly, expand '$(y - 7)^2$' in one step.

Expand.

1. $(x - 2)^2$
2. $(y + 4)^2$
3. $(z - 8)^2$
4. $(x - 1)^2$
5. $(p - 7)^2$
6. $(y + 9)^2$
7. $3(x + 5)^2$
8. $-5(x - 4)^2$
9. $-(x - 4)^2$
10. $(x - \frac{3}{2})^2$ [Answer. $x^2 - 3x + \frac{9}{4}$]
11. $(x - \frac{1}{2})^2$
12. $(x + \frac{3}{2})^2$
13. $(y + \frac{5}{2})^2$
14. $(z - \frac{7}{2})^2$

15. $(x - \frac{1}{4})^2$ 16. $(x + \frac{5}{4})^2$ 17. $(x - \frac{k}{2})^2$ 18. $(y + \frac{m}{2})^2$

19. $(x - \frac{m}{n})^2$ 20. $(x - \frac{3}{m})^2$ 21. $(y - \frac{5}{2n})^2$ 22. $(x + \frac{b}{2a})^2$

B. Each of the following is a quadratic expression in 'x'. Tell which of them are perfect squares. [A perfect square is the expression you get when you expand an expression like most of those in Part A.]

Sample 1. $x^2 - 8x + 16$

Solution. This is a perfect square because

$$\forall_x \ (x - 4)^2 = x^2 - 8x + 16.$$

Sample 2. $x^2 - 3x + (\frac{3}{2})^2$

Solution. This is a perfect square because

$$\forall_x \ (x - \frac{3}{2})^2 = x^2 - 3x + (\frac{3}{2})^2.$$

1. $x^2 + 6x + 9$ 2. $x^2 + 4x + 16$ 3. $x^2 - 4x + 4$

4. $x^2 - 10x - 25$ 5. $x^2 - 20x + 100$ 6. $x^2 - 12x + 36$

7. $x^2 + 5x + (\frac{5}{2})^2$ 8. $x^2 + 7x + \frac{7}{2}$ 9. $x^2 - 3x + \frac{9}{4}$

10. $x^2 - 11x - \frac{121}{4}$ 11. $x^2 + 2ax + a^2$ 12. $x^2 - 4bx + 2b^2$

13. $x^2 - 8cx + 16c^2$ 14. $x^2 + 9mx + (\frac{9m}{2})^2$ 15. $x^2 + \frac{m}{n}x + \frac{m^2}{n^2}$

16. $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$

C. Each of the following is a quadratic expression in 'x'. Your job is to tack something onto it so that the resulting expression is a perfect square, and to prove that it is a perfect square.

Sample. $x^2 + 11x$

Solution. $x^2 + 11x + (\frac{11}{2})^2$

Proof. $\forall_x \ (x + \frac{11}{2})^2 = x^2 + 11x + (\frac{11}{2})^2$

1. $x^2 + 8x$ 2. $x^2 + 2x$ 3. $x^2 - 6x$ 4. $x^2 - 12x$

5. $x^2 + 20x$ 6. $x^2 + 1000x$ 7. $x^2 - 500x$ 8. $x^2 + 4bx$

9. $x^2 - 6kx$ 10. $x^2 + 3x$ 11. $x^2 + 9x$ 12. $x^2 - 7x$

13. $x^2 - \frac{1}{2}x$ 14. $x^2 - kx$ 15. $x^2 - \frac{1}{k}x$ 16. $x^2 - \frac{4x}{k}$

17. $x^2 - \frac{3x}{k}$ 18. $x^2 + \frac{bx}{a}$

D. Transform each of the following quadratic expressions in 'x' by completing the square, and simplifying.

Sample 1. $x^2 + 3x + 7$

Solution. '$x^2 + 3x$' can be made into a perfect square by tacking on a '$+ (\frac{3}{2})^2$'. Suppose we just write:

$$x^2 + 3x + (\frac{3}{2})^2 + 7$$

This expression is not equivalent to the given one. But, we can get an equivalent expression by writing:

$$x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 + 7$$

This last expression is equivalent to:

$$(x + \frac{3}{2})^2 - (\frac{3}{2})^2 + 7$$

And, this is the expression in which the square has been completed. We simplify it:

$$(x + \frac{3}{2})^2 - \frac{9}{4} + 7$$

$$= (x + \frac{3}{2})^2 - \frac{9}{4} + \frac{28}{4}$$

$$= (x + \frac{3}{2})^2 + \frac{19}{4}$$

Answer. $(x + \frac{3}{2})^2 + \frac{19}{4}$

1. $x^2 + 6x + 2$ 2. $x^2 + 12x - 3$ 3. $x^2 - 8x + 9$

4. $x^2 - 16x + 30$ 5. $x^2 + 10x + 11$ 6. $x^2 - 4x + 2$

7. $x^2 + x - 7$ 8. $x^2 - x$ 9. $x^2 + 5x + 4$

10. $x^2 - 7x - 2$ 11. $x^2 + \frac{1}{2}x + 5$ 12. $x^2 - \frac{1}{3}x + 1$

13. $x^2 + 2mx + n$ 14. $x^2 - 4bx + d$ 15. $x^2 - \frac{x}{k} + \frac{c}{k}$

16. $x^2 + \frac{bx}{a} + \frac{c}{a}$

Sample 2. $3x^2 + 6x + 5$

Solution. $3x^2 + 6x + 5$

$$= 3(x^2 + 2x) + 5$$

$$= 3(x^2 + 2x + 1 - 1) + 5$$

$$= 3[(x + 1)^2 - 1] + 5$$

$$= 3(x + 1)^2 - 3 + 5$$

$$= 3(x + 1)^2 + 2$$

17. $2x^2 - 20x + 7$
18. $5x^2 + 20x - 4$
19. $3x^2 + 12x - 1$
20. $2x^2 + x - 9$
21. $2x^2 - 3x + 4$
22. $-5x^2 + 5x - 4$
23. $ax^2 + 5ax + 7$
24. $ax^2 + bx + c$

E. For each function, find the ordered pair in it whose second component is the extreme value of the function, and tell whether the extreme value is a minimum or a maximum.

Sample. $\{(x, y): y = 3(x - 4)^2 + 7\}$

Solution. (4, 7); minimum

1. $\{(x, y): y = x^2 + 9x - 8\}$
2. $\{(x, y): y = -x^2 - 9x + 8\}$
3. $\{(x, y): y = 5x^2 - 2x + 7\}$
4. $\{(x, y): y = 9 + 5x - 7x^2\}$
5. $\{(a, b): b = 1 + a + 3a^2\}$
6. $\{(u, v): u^2 - v = 2u - 1\}$
7. $\{(x, y): x \geq 2 \text{ and } y = x^2 + x + 7\}$
8. $\{(x, y): x > 2 \text{ and } y = x^2 + x + 7\}$
9. $\{(x, y): -1 \leq x \leq 2 \text{ and } y = x^2 + x + 7\}$

F. 1. Solve the fencing problem given on page 5-166, without drawing a graph, by finding the extreme point of the quadratic function

$$\{(x, y): y = (120 - 2x)x\}.$$

2. A rectangular field is to be fenced off beside a river. No fence is needed along the river bank. What are the dimensions of the field of largest area which can be fenced off using 100 yards of fencing?

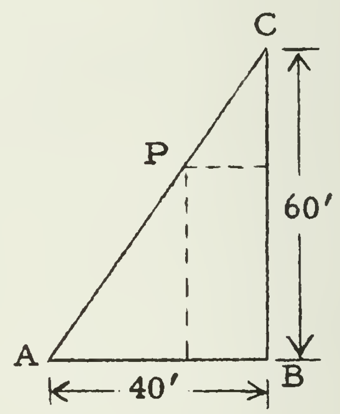
3. A man has 600 yards of fencing which he is going to use to enclose a rectangular field and to divide it down the middle with a fence parallel to one side. What are the dimensions of the largest field he can enclose?

4. A long rectangular sheet of metal 10 inches wide is to be made into a trough by bending up the sides. How deep should the trough be to carry the most water? [Cross section:]

5. A small orchard now has 60 trees; it yields, on the average, 400 apples per tree. For each additional tree planted in this orchard the average yield per tree will be reduced by approximately 6 apples. How many trees will give the largest crop of apples for this orchard?

6. Prove that, of all rectangles with a given perimeter p, those which are squares have the greatest area. [Hint. If the width-measure of a rectangle of perimeter p is x then its length-measure is ______.]

7. A man wants to build a house, having a rectangular foundation, on a lot whose shape is such that, taking account of building restrictions, the foundation must not extend outside a right triangular region, as shown:

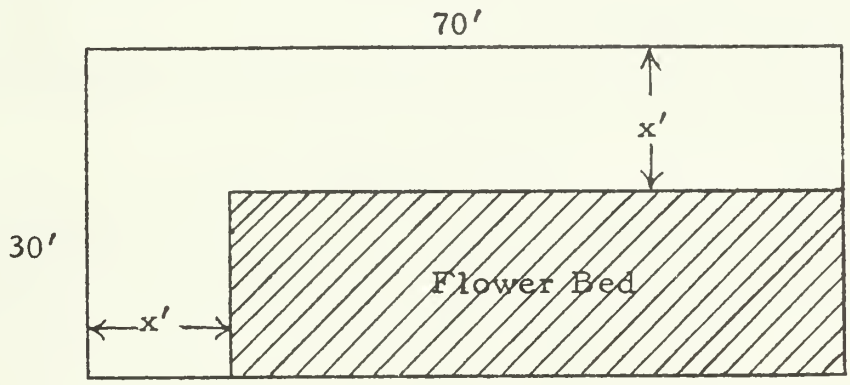


If one corner of the foundation is at B, how far should each of the adjacent corners be from B to make the area enclosed by the foundation as large as possible?
[Hint. If you think of A as the origin of the number plane and of B as a point on the positive half of the x-axis, what can you say about the slopes of segments $\overline{AP}$ and $\overline{AC}$?]

8. A potato grower wishes to ship as early as possible in the season in order to sell at the best price. If he ships July 1, he can ship 6 tons at a profit of \$2.00 per ton. He estimates that by waiting he can add 3 tons per week to his shipment, but that the profit will be reduced $\frac{1}{3}$ dollar per ton per week. When should he ship for maximum profit?

[Supplementary exercises are in Part U, page 5-272.]

5.10 Quadratic equations. --a gardener is planning a flower bed which is to take up half the area of a 30′ by 70′ rectangular plot of land. The flower bed is to be rectangular in shape and situated in one corner of the plot in such a way that the rest of the plot can be paved to provide a sidewalk of uniform width on both sides of the bed.
How wide should the sidewalk be?



If the sidewalk is x feet wide, the dimensions of the flower bed are 70 - x feet and 30 - x feet, and its area is (70 - x)(30 - x) square feet. So, we are looking for a number x between 0 and 30 such that

$$(70 - x)(30 - x) = \frac{1}{2} \cdot 2100.$$

Let's try to solve this equation. Expand the left side.

$$2100 - 100x + x^2 = 1050$$

$$x^2 - 100x + 1050 = 0$$

This is a quadratic equation, and as you may recall from Unit 3, since the right side is '0', we should try to factor the left side. This turns out to be a difficult task. Try it and see.

Let's look for another method to give us factors, a method which is more systematic the usual trial-and-error procedure. The left side is a quadratic expression in 'x'. Let's complete the square.

$$\begin{aligned} & x^2 - 100x + 1050 \\ = \ & x^2 - 100x + 50^2 - 50^2 + 1050 \\ = \ & (x - 50)^2 - 2500 + 1050 \\ = \ & (x - 50)^2 - 1450 \end{aligned}$$

So, the given equation is equivalent to:

$$(*) \qquad (x - 50)^2 - 1450 = 0$$

Now, we could solve this equation graphically by considering the quadratic function $\{(x, y): y = (x - 50)^2 - 1450\}$, graphing it, and finding the places where the graph crosses the horizontal axis. These are the graphs of points with second component 0. So, their first components are numbers which satisfy (*).

Another way to solve (*) is to recall how you would solve an equation like:

$$y^2 - 4^2 = 0$$

The left side is easily factored. So,

$$(y - 4)(y + 4) = 0,$$

$$y - 4 = 0 \text{ or } y + 4 = 0,$$

$$y = 4 \text{ or } y = -4.$$

Equation (*) can be handled the same way. We note that $(x - 50)^2$ is the square of $x - 50$, and that 1450 is the square of $\sqrt{1450}$. Hence, (*) is equivalent to:

$$(x - 50)^2 - (\sqrt{1450})^2 = 0,$$

$$[(x - 50) - \sqrt{1450}][(x - 50) + \sqrt{1450}] = 0,$$

$$x - 50 - \sqrt{1450} = 0 \text{ or } x - 50 + \sqrt{1450} = 0,$$

$$x = 50 + \sqrt{1450} \text{ or } x = 50 - \sqrt{1450}$$

So, the roots of (*) are $50 + \sqrt{1450}$ and $50 - \sqrt{1450}$. Since $30 < \sqrt{1450} < 40$, the larger root is between 80 and 90, and the smaller one is between 10 and 20. The larger root doesn't meet the conditions of our problem since we are looking for a number between 0 and 30. Hence, the sidewalk should be $50 - \sqrt{1450}$ feet wide.

The numeral '$50 - \sqrt{1450}$' is not very handy for a gardener since tape measures are not marked this way. The gardener would be satisfied with a rational number approximation to $50 - \sqrt{1450}$. First, we find a rational number approximation to $\sqrt{1450}$. Since 1450 is 14.5×10^2, $\sqrt{1450} = 10\sqrt{14.5}$. The table on page 5-219 gives 3.742 and 3.873 as

rational approximations to $\sqrt{14}$ and $\sqrt{15}$. So, $\sqrt{14.5}$ is between 3.742 and 3.873. Hence, $37.42 < \sqrt{1450} < 38.73$. We also see from the table that 38^2 is 1444, which is less than 1450. So, $\sqrt{1450}$ is very close to 38. $50 - 38 = 12$. So, the sidewalk should be about 12 feet wide.

The technique of completing the square is one which can be used in solving all quadratic equations in one variable. Of course, some quadratic equations involve quadratic expressions whose factors can be found readily by inspection. For example, we wouldn't take the trouble to complete the square in solving:

$$x^2 - 5x + 6 = 0$$

since it is easier to factor '$x^2 - 5x + 6$' by trial-and-error. [What are the roots?] But, we <u>could</u> find the factors by completing the square:

$$x^2 - 5x + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 6 = 0$$

$$(x - \frac{5}{2})^2 - \frac{25}{4} + 6 = 0$$

$$(x - \frac{5}{2})^2 - \frac{1}{4} = 0$$

$$(x - \frac{5}{2})^2 - (\frac{1}{2})^2 = 0$$

$$[(x - \frac{5}{2}) - \frac{1}{2}][(x - \frac{5}{2}) + \frac{1}{2}] = 0$$

$$x - \frac{5}{2} = \frac{1}{2} \text{ or } x - \frac{5}{2} = -\frac{1}{2}$$

$$x = 3 \text{ or } x = 2$$

Practice using the method of completing the square in solving the quadratic equations '$x^2 - 8x - 20 = 0$' and '$x^2 + 3x + 2 = 0$'.

In solving a quadratic equation it is usually helpful to transform it to one of the form:

$$(*) \qquad ax^2 + bx + c = 0$$

This is called <u>the standard form of a quadratic equation in one variable</u>. An equation in one variable is a quadratic equation if and only if it can be transformed to an equation of the form of (*), in which the value of 'a' is not zero.

Example 1. Solve: $2x(x + 3) = 9(x + 1) - 2$

Solution. First, transform it to an equation of standard form.

$$2x(x + 3) = 9(x + 1) - 2$$
$$2x^2 + 6x = 9x + 9 - 2$$
$$2x^2 - 3x - 7 = 0$$

The factors are not apparent by inspection. So, we complete the square. First, multiply by $\frac{1}{2}$ to make it a little easier to complete the square.

$$\frac{1}{2}(2x^2 - 3x - 7) = 0 \cdot \frac{1}{2}$$
$$x^2 - \frac{3}{2}x - \frac{7}{2} = 0$$

Now, complete the square.

$$x^2 - \frac{3}{2}x + (\frac{3}{4})^2 - (\frac{3}{4})^2 - \frac{7}{2} = 0$$
$$(x - \frac{3}{4})^2 - \frac{9}{16} - \frac{56}{16} = 0$$
$$(x - \frac{3}{4})^2 - \frac{65}{16} = 0$$
$$[(x - \frac{3}{4}) - \sqrt{\frac{65}{16}}\,][(x - \frac{3}{4}) + \sqrt{\frac{65}{16}}\,] = 0$$
$$x - \frac{3}{4} = \sqrt{\frac{65}{16}} \text{ or } x - \frac{3}{4} = -\sqrt{\frac{65}{16}}$$
$$x = \frac{3}{4} + \sqrt{\frac{65}{16}} \text{ or } x = \frac{3}{4} - \sqrt{\frac{65}{16}}$$

The roots are $\frac{3}{4} + \sqrt{\frac{65}{16}}$ and $\frac{3}{4} - \sqrt{\frac{65}{16}}$. Can these expressions for the roots be simplified? Yes, this principle for real numbers:

$$\forall_{x \geq 0}\ \forall_{y > 0}\ \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

assures us that $\frac{3}{4} + \sqrt{\frac{65}{16}} = \frac{3}{4} + \frac{\sqrt{65}}{\sqrt{16}} = \frac{3}{4} + \frac{\sqrt{65}}{4} = \frac{3 + \sqrt{65}}{4}$.

So, the roots are $\frac{3 + \sqrt{65}}{4}$ and $\frac{3 - \sqrt{65}}{4}$.

Example 2. Solve: $5x - \frac{2}{x} + 10 = 0$

Solution. Transform to an equation in standard form.

$$5x - \frac{2}{x} + 10 = 0$$

$$x(5x - \frac{2}{x} + 10) = 0 \cdot x$$

$$5x^2 - 2 + 10x = 0$$

$$5x^2 + 10x - 2 = 0$$

[All of the roots of the given equation are roots of this last equation. However, if the last equation has 0 as a root, this root is not a root of the given equation. Why?] Now, complete the square. First, multiply by $\frac{1}{5}$.

$$x^2 + 2x - \frac{2}{5} = 0$$

$$(x + 1)^2 - 1 - \frac{2}{5} = 0$$

$$(x + 1)^2 - \frac{7}{5} = 0$$

$$[(x + 1) - \sqrt{\frac{7}{5}}][(x + 1) + \sqrt{\frac{7}{5}}] = 0$$

$$x + 1 = \sqrt{\frac{7}{5}} \text{ or } x + 1 = -\sqrt{\frac{7}{5}}$$

$$x = -1 + \sqrt{\frac{7}{5}} \text{ or } x = -1 - \sqrt{\frac{7}{5}}$$

The roots are $-1 + \sqrt{\frac{7}{5}}$ and $-1 - \sqrt{\frac{7}{5}}$.

EXERCISES

A. Solve these equations by completing the square.

1. $x^2 + 6x - 23 = 0$
2. $x^2 + 8x - 5 = 0$
3. $x^2 + 5x - 1 = 0$
4. $x^2 - 9x + 4 = 0$
5. $x^2 - 2x = 7$
6. $x^2 + x - 9 = 3$
7. $2x^2 + 9x - 8 = 0$
8. $2x^2 - x - 3 = 0$
9. $3x^2 - 2x - 1 = 5x$
10. $x^2 + 4 = 8x$
11. $2(3 - x) = 3x^2$
12. $4x^2 + x + 8 = 11 - x^2$
13. $x^2 + 8x + 16 = 0$
14. $x^2 - 10x + 25 = 0$

B. For each exercise, find the points [if any] in the intersection of the given function with the x-axis.

Sample. $\{(x, y): y = x^2 - 10x + 26\}$

Solution. We could do this problem by graphing the function and estimating the first components of the points in the intersection of the function with the x-axis. But, it is much easier just to find the roots of the equation:

$$x^2 - 10x + 26 = 0$$

We solve by completing the square.

$$(x - 5)^2 - 25 + 26 = 0$$
$$(x - 5)^2 + 1 = 0$$

Now, '$(x - 5)^2 + 1$' cannot be factored as in the other cases because it is not an expression of the form '$A^2 - B^2$'. In fact, since $(x - 5)^2$ is nonnegative for each x, there is no real number x such that $(x - 5)^2 + 1 = 0$. Hence, the quadratic equation has no roots, and the intersection of the function with the x-axis is $\emptyset$.

1. $\{(x, y): y = x^2 + 14x - 140\}$
2. $\{(x, y): y = x^2 - 20x + 96\}$
3. $\{(x, y): y = 3x - 12\}$
4. $\{(x, y): y = (x - 5)(x + 7)\}$
5. $\{(x, y): y = 2x^2 - 7x - 15\}$
6. $\{(x, y): y = x^2 - 8x + 16\}$
7. $\{(x, y): y = 5x^2 - 20\}$
8. $\{(x, y): y = 5x^2 + 20\}$
9. $\{(x, y): y = (x - 3)^2 - 4\}$
10. $\{(x, y): y = (x + 1)(x - 2)^2\}$

☆11. $\{(x, y): y = x^2 + px + q\}$

☆12. $\{(x, y): y = ax^2 + bx + c\}$, $[a \neq 0]$

THE QUADRATIC FORMULA

Since a quadratic equation in one variable can be transformed to standard form:

$$ax^2 + bx + c = 0, \ [a \neq 0]$$

the solution set of the quadratic equation is completely determined by the numbers a, b, and c. Similarly, since each linear equation in one variable can be transformed to standard form:

$$(*) \qquad ax + b = 0, \ [a \neq 0]$$

the solution set of a linear equation is completely determined by the numbers a and b. In fact, for each $a \neq 0$, for each b,

$$(**) \qquad \{x:\ ax + b = 0\} = \{-\frac{b}{a}\}.$$

It would be helpful if we could find a similar result for quadratic equations. This would then give us a quick way of finding the solution set of a quadratic equation without bothering to hunt for factors or to complete the square.

Equation (**) was derived from (*) simply by solving (*) for 'x'. So, let's try to solve:

$$ax^2 + bx + c = 0$$

for 'x'.

We solve by the method of completing the square.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + (-\frac{b^2}{4a^2} + \frac{c}{a}) = 0$$

$$(x + \frac{b}{2a})^2 + \frac{-b^2 + 4ac}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 + \frac{-(b^2 - 4ac)}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 + -\frac{b^2 - 4ac}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

Now, in order to be able to factor, we must be sure that $\frac{b^2 - 4ac}{4a^2}$ is the square of a real number. Hence, it must be nonnegative. [Or, to put it another way, if $\frac{b^2 - 4ac}{4a^2}$ is negative, since $(x + \frac{b}{2a})^2$ is nonnegative, there is no number x such that $(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$.] We know that $4a^2$ is positive. So, $\frac{b^2 - 4ac}{4a^2}$ is nonnegative if and only if $b^2 - 4ac$ is

nonnegative. Thus, we can factor if we suppose that $b^2 - 4ac \geq 0$. Let's suppose this, and continue solving for 'x'.

$$(x + \frac{b}{2a})^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$$

$$\left[(x + \frac{b}{2a}) - \sqrt{\frac{b^2 - 4ac}{4a^2}}\right]\left[(x + \frac{b}{2a}) + \sqrt{\frac{b^2 - 4ac}{4a^2}}\right] = 0$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Now, since $b^2 - 4ac \geq 0$ and $4a^2 > 0$,

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{|2a|}.$$

Thus, the roots are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{|2a|} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{|2a|}.$$

[If a is positive then $|2a| = 2a$, and if a is negative, $|2a| = -2a$.]

So, we have proved the theorem:

$$\forall_{a \neq 0} \forall_b \forall_c \text{ such that } b^2 - 4ac \geq 0,$$

$$\{x\colon ax^2 + bx + c = 0\} = \{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\}$$

[What happened to the '$|2a|$'?]

Sometimes, in order to be brief, we write:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and call this the quadratic formula. The symbol '±' [read as 'plus or minus'] reminds you that <u>each</u> of the numbers

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

is a root of the equation '$ax^2 + bx + c = 0$'.

Let's use the quadratic formula in solving the equation:

$$x(2x + 3) = 3(1 - x)$$

First, we transform this equation into one of standard form.

$$2x^2 + 3x = 3 - 3x$$

$$2x^2 + 6x - 3 = 0$$

Here, a = 2, b = 6, and c = −3. So, the roots can be obtained from the quadratic formula:

$$\frac{-6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot -3}}{2 \cdot 2}$$

$$= \frac{-6 \pm \sqrt{60}}{4}$$

$$= \frac{-6 \pm \sqrt{4 \cdot 15}}{4}$$

$$= \frac{-6 \pm \sqrt{4} \cdot \sqrt{15}}{4}$$

$$= \frac{-6 \pm 2\sqrt{15}}{4}$$

$$= \frac{2(-3 \pm \sqrt{15})}{4}$$

$$= \frac{-3 \pm \sqrt{15}}{2}$$

The roots are $\frac{-3 + \sqrt{15}}{2}$ and $\frac{-3 - \sqrt{15}}{2}$.

Let's try another quadratic equation.

$$x^2 - 5x + 7 = 0$$

This equation is in standard form, and a = 1, b = −5, and c = 7. If we apply the quadratic formula and simplify, we get:

$$\frac{5 \pm \sqrt{-3}}{2}$$

But, '$\sqrt{-3}$' doesn't stand for a real number. Recall, from our derivation of the quadratic formula, that the given equation has real number roots just if $b^2 - 4ac \geq 0$. In the case at hand, $b^2 - 4ac = -3$ and $-3 \not\geq 0$. So, no real number satisfies the given equation.

EXERCISES

A. Use the quadratic formula to solve each equation. If no real number satisfies it, say so.

1. $2x^2 - 7x - 5 = 0$
2. $3x^2 + x - 7 = 0$
3. $5x^2 + x = 1$
4. $2x(2x + 7) = 2x - 3$
5. $6x - 8x^2 + 7 = 0$
6. $8 + 3y - 5y^2 = 0$
7. $3n(2 - n) = 5n + 6$
8. $2x(5 + x) = x^2 + x(4 + x)$
9. $5p^2 + 2p = 0$
10. $5 - 11x^2 = 0$
11. $2x + \frac{3}{x} + 6 = 0$
12. $5x + \frac{7}{x} = 0$
13. $(1 - x)^2 + 3(1 - x) - 5 = 0$
14. $3(\frac{1}{x})^2 + \frac{2}{x} - 15 = 0$

☆15. $(\sqrt{x})^2 + 2\sqrt{x} - 4 = 0$

☆16. $\sqrt{x} - x - 3 = 0$

B. For each quadratic equation, tell whether it has roots. If it does, tell whether it has two roots or one root.

Sample. $x^2 + 6x + 9 = 0$

Solution. By the quadratic formula,

$$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 1 \cdot 9}}{2}$$

$$= \frac{-6 \pm \sqrt{0}}{2}.$$

Since $-6 + 0 = -6$ and $-6 - 0 = -6$, the equation has exactly one root, the real number -3. [Do you see a quick way to determine whether a quadratic equation has 0, 1, or 2 real number roots?]

1. $x^2 - 8x + 16 = 0$
2. $x^2 - x - 3 = 0$
3. $x^2 - 7 = 0$
4. $8x^2 + 24x + 9 = 0$
5. $8x^2 - 24x + 9 = 0$
6. $(x - 3)^2 - 5 = 0$
7. $(x - 3)^2 = 0$
8. $(x - 3)^2 + 5 = 0$
9. $5 - (3 - x)^2 = 0$

A quadratic equation like the one in the Sample is sometimes said to have <u>a double root</u>. Then, if you count a double root as two roots, you can say that each quadratic equation whose solution set is not $\emptyset$ has two roots.

C. Solve these equations. [If the roots are irrational numbers, give rational approximations to them which are correct to the nearest tenth.]

Sample. $2x^2 - 5x - 8 = 0$

Solution. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot -8}}{2 \cdot 2}$$

$$\frac{5 \pm \sqrt{25 + 64}}{4}$$

The roots are $\frac{5 + \sqrt{89}}{4}$ and $\frac{5 - \sqrt{89}}{4}$.

The table on page 5-219 tells us that $\sqrt{89}$ is 9.434 correct to the nearest thousandth.

$$\frac{5 + 9.434}{4} = \frac{14.434}{4} = 3.6085$$

$$\frac{5 - 9.434}{4} = \frac{-4.434}{4} = -1.1085$$

So, rational approximations correct to the nearest tenth are 3.6 and -1.1.

1. $x^2 - 5x - 7 = 0$ 2. $8 = 5x + x^2$ 3. $3x^2 + 2x - 5 = 0$

4. $3x(x - 1) = 1$ 5. $11 - x^2 = 0$ 6. $3(x - 2)^2 = 13(x - 2) + 2$

☆ D. Write a quadratic equation in 'x' in standard form whose solution set is

1. $\{3, 4\}$ 2. $\{5\}$ 3. $\{0\}$ 4. $\{\sqrt{10}, -\sqrt{10}\}$

5. $\{8, -8\}$ 6. $\{\frac{2}{3}, -\frac{3}{4}\}$ 7. $\{r_1, r_2\}$ 8. $\{2 - \sqrt{5}, 2 + \sqrt{5}\}$

☆ E. Solve these equations.

1. $x^2 + x\sqrt{2} - 4 = 0$ 2. $x^2 - x\sqrt{3} - \frac{3}{4} = 0$

☆ F. Graph these sentences.

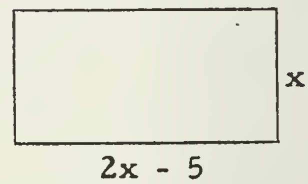
1. $y \geq x^2$ 2. $y \leq (x - 3)^2 - 5$ 3. $x < y^2 - y + 2$

4. $y \geq (x - 2)(2x + 5)$ 5. $y = |(x - 2)(x - 4)|$

G. In solving the following problems you may find it convenient to use quadratic equations.

Sample 1. The length of a certain rectangle is 5 inches less than twice its width. The area is 75 square inches. Find the number of inches in the width and the number of inches in the length.

Solution. Suppose the width is x inches. Then, the length is $2x - 5$ inches and, since the area is 75 square inches,



$$(*) \qquad x(2x - 5) = 75.$$

Now, $x(2x - 5) = 75$ if and only if $2x^2 - 5x - 75 = 0$. And, $2x^2 - 5x - 75 = 0$ if and only if $x = \frac{5 \pm \sqrt{25 + 600}}{4}$.

$$\frac{5 \pm \sqrt{25 + 600}}{4} = \frac{5 \pm \sqrt{625}}{4} = \frac{5 \pm 25}{4}$$

Hence, the roots of (*) are 7.5 and −5. So, the width is 7.5 inches [Why not −5?] and the length is $2 \cdot 7.5 - 5$, or 10 inches. [Notice that (*) does not completely characterize the measure of the width of the rectangle. You also need the sentence '$x > 5/2$'. Why?]

Check. $10 = 2 \cdot 7.5 - 5$; $10 \cdot 7.5 = 75$

Sample 2. A class of students wanted to buy a \$12 gift for the school principal [Zabranchburg High, where else?]. They decided on an amount to assess each class member. Six persons in another class heard of the plan and wanted to join in buying the gift. The assessment for each person was then reduced by 10 cents. How many persons were in the class?

Solution. Suppose there were x persons in the class. Then each of these x persons was originally assessed $\frac{12}{x}$ dollars. When they were joined by the other 6, the assessment dropped to $\frac{12}{x} - \frac{1}{10}$ dollars. So, $x + 6$ people pay a total of $(x + 6)(\frac{12}{x} - \frac{1}{10})$ dollars. Thus, we seek a number x [a positive integer] such that

$$(x + 6)\left(\frac{12}{x} - \frac{1}{10}\right) = 12.$$

$$(x + 6)\left(\frac{120 - x}{10x}\right) = 12$$

$$(x + 6)(120 - x) = 120x$$

$$720 + 114x - x^2 = 120x$$

$$x^2 + 6x - 720 = 0$$

$$(x - 24)(x + 30) = 0$$

$$x = 24 \text{ or } x = -30$$

So, there were 24 members in the class.

Check. Each of 24 students contributes 50 cents to collect \$12, and each of 30 students contributes 40 cents to collect the same amount.

1. Find three consecutive real integers such that the sum of their squares is 302. $[(x - 1)^2 + x^2 + (x + 1)^2 = 302]$

2. Find four consecutive odd real integers such that the sum of their squares is 36.

3. The inch-perimeter of a given rectangle is 38, and the area of the rectangle is 78 square inches. Find the width and the length.

4. Find the dimensions of the largest rectangular field which can be fenced with 1000 feet of fencing.

5. Find two numbers whose difference is 1, and the sum of whose reciprocals is $\frac{40}{21}$.

6. If you add 12 to the number of square inches in the area of a square, you get the number of inches in the perimeter. Find the length of a side of the square.

7. A hiker walked 6 miles. To have walked this distance in $\frac{1}{2}$ hour less time, he would have had to walk 2 miles faster per hour. Find his rate and how long he hiked.

8. The number named by the numerator of a certain fraction exceeds the denominator-number by 2, and the fraction stands for a number which exceeds its reciprocal by $\frac{24}{35}$. Find the fraction.

[Supplementary exercises are in Part V, pages 5-273 through 5-274.]

EXPLORATION EXERCISES

A. 1. Graph these equations on the same chart.

(1) $5x - 6y + 30 = 0$

(2) $10x - 3y - 30 = 0$

2. Graph these equations on the same chart used in Exercise 1.

(3) $6[5x - 6y + 30] - 2[10x - 3y - 30] = 0$

(4) $4[5x - 6y + 30] - 3[10x - 3y - 30] = 0$

3. Graph these equations on the same chart used in Exercise 1.

(5) $2[5x - 6y + 30] - 1[10x - 3y - 30] = 0$

(6) $1[5x - 6y + 30] - 2[10x - 3y - 30] = 0$

4. What ordered pair (x, y) satisfies both (1) and (2)? Both (3) and (4)? Both (5) and (6)?

5. For each equation given below, find one ordered pair (x, y) which satisfies it.

(a) $9[5x - 6y + 30] - 7[10x - 3y - 30] = 0$

(b) $9[5x - 6y + 30] + 7[10x - 3y - 30] = 0$

(c) $84[5x - 6y + 30] - 257[10x - 3y - 30] = 0$

(d) $-34[5x - 6y + 30] - 2172[10x - 3y - 30] = 0$

(e) $\frac{1}{2}[5x - 6y + 30] + \frac{1}{3}[10x - 3y - 30] = 0$

B. Pick out those relations from the list below whose graphs are either vertical or horizontal straight lines, and tell which.

1. $\{(x, y): 3[5x + 7y - 8] - 5[3x - 2y + 12] = 0\}$

2. $\{(x, y): 9[2x - y + 3] + [57x + 9y - 81] = 0\}$

3. $\{(a, b): 7[a - 3b - 12] - [7a + 15b - 78] = 0\}$

4. $\{(a, b): 15[2a - 6b - 9] + 7[3a - 5b + 2] = 0\}$

5. $\{(m, n): 85[3m - 17n + 71] - 3[85m + 52n - 62] = 0\}$

6. $\{(p, q): [p - 3q + 5] + [q - p + 18] = 0\}$

C. For each exercise, put numerals in the frames so that the resulting expression names a relation whose graph is a vertical straight line.

1. $\{(x, y): \square[3x - 5y + 4] + \bigcirc[2x + 7y - 5] = 0\}$

2. $\{(x, y): \square[x + 7y - 3] + \bigcirc[5x + 3y - 2] = 0\}$

3. $\{(x, y): \square[\frac{1}{4}x - 2y + 7] + \bigcirc[2x - 3y - 9] = 0\}$

4. $\{(x, y): \square[2y + 7x - 3] + \bigcirc[8 + 5y - 4x] = 0\}$

5. $\{(x, y): \square[5x - 2y + 5] + \bigcirc[9x + 2y - 3] = 0\}$

6. $\{(x, y): \square[5x - 2y + 5] + \bigcirc[9x - 2y - 3] = 0\}$

D. Repeat Part C, but get horizontal straight lines.

E. 1. Find numbers x and y such that

$$3x - 5y + 4 = 0$$

and $$2x + 7y - 5 = 0.$$

[Refer to Exercise 1 of Part C and Exercise 1 of Part D.]

2. Find numbers x and y such that

$$x + 7y - 3 = 0$$

and $$5x + 3y - 2 = 0.$$

[Refer to Exercise 2 of Parts C and D.]

3. Find an ordered pair (x, y) which satisfies both of the equations:

$$\frac{1}{4}x - 2y + 7 = 0$$

and: $$2x - 3y - 9 = 0$$

4. Find a member of the intersection of

$\{(x, y): 2y + 7x - 3 = 0\}$ and $\{(x, y): 8 + 5y - 4x = 0\}$.

5. $\{(x, y): 5x - 2y + 5 = 0\} \cap \{(x, y): 9x + 2y - 3 = 0\} = \{\underline{\qquad\qquad}\}$

6. Find an ordered pair which belongs to

$\{(x, y): 5x - 2y + 5 = 0 \text{ and } 9x - 2y - 3 = 0\}$.

5.11 <u>Systems of equations</u>. --The preceding Exploration Exercises dealt with the problem of solving a system of two linear equations in two variables. In an earlier discussion [page 5-191] we said that a linear equation in one variable 'x' is an equation which can be transformed to one of the [standard] form:

$$ax + b = 0, \quad [a \neq 0]$$

<u>A linear equation in two variables 'x' and 'y'</u> is one which can be transformed into an equation of the [standard] form:

$$ax + by + c = 0, \quad [a \neq 0 \text{ or } b \neq 0]$$

[Do you see that the condition '$a \neq 0$ or $b \neq 0$' permits you to consider a linear equation in <u>one</u> variable as a linear equation in two variables? For example, the equation '$2y + 7 = 0$' can be thought of as an abbreviation for '$0x + 2y + 7 = 0$'. But, you must be careful to note that the solution set of such an equation, when you think of it as an equation in two variables, is a set of ordered pairs and so is different from the solution set of the same equation when you consider it as an equation in one variable. For example, $\{(x, y)\colon\ 2y + 7 = 0\}$ is the set of all ordered pairs of real numbers which have $-\frac{7}{2}$ for second component. But, $\{y\colon\ 2y + 7 = 0\} = \{-\frac{7}{2}\}$.]

A pair of linear equations in two variables 'x' and 'y' is called <u>a system of two linear equations in two variables 'x' and 'y'</u>. To solve a system of two equations is to find the ordered pairs (x, y) which satisfy both equations in the system. So, to solve the system of linear equations in 'x' and 'y':

$$\left.\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}\right\}$$

is to find the members of

$$\{(x, y)\colon\ ax + by + c = 0\} \cap \{(x, y)\colon\ a'x + b'y + c' = 0\},$$

or, in other words, to find the members of:

$$\{(x, y)\colon\ ax + by + c = 0 \text{ and } a'x + b'y + c' = 0\}$$

The latter is the <u>solution set</u> of the system, and each member of it is called <u>a solution of the system</u>.

You know that, for each a and b, $\{(x, y): ax + by + c = 0\}$ is a linear function if and only if $a \neq 0 \neq b$. So, you can use your knowledge of linear functions in solving many systems of linear equations in two variables. For example, consider the following system:

$$\left.\begin{array}{ll} (1) \qquad & 3x + y - 14 = 0 \\ (2) \qquad & 5x - y - 18 = 0 \end{array}\right\}$$

Solving each equation for 'y' we obtain the equivalent equations:

$$\begin{array}{ll} (1') \qquad & y = -3x + 14 \\ (2') \qquad & y = 5x - 18, \end{array}$$

which are clearly defining equations for linear functions. So, to solve the system is to find ordered pairs which belong to both functions. A number x is the first component of such an ordered pair if and only if

$$-3x + 14 = 5x - 18 \qquad \text{[Explain].}$$

The root of this equation is 4, and substituting '4' for 'x' in any of the equations (1), (2), (1′), and (2′) gives 2 as the corresponding value of 'y'. So, the single solution of the system consisting of (1) and (2) is (4, 2). So, the solution set is $\{(4, 2)\}$.

Now, the foregoing is an excellent technique for solving a system whose equations are easy to solve for 'y' [or for 'x']. But, consider the following system.

$$\left.\begin{array}{ll} (3) \qquad & 5x - 3y - 11 = 0 \\ (4) \qquad & 7x - 6y - 10 = 0 \end{array}\right\}$$

Solving these equations for 'y' [or for 'x'] involves messy fractions. There is a better way to solve the system, and this way was illustrated in the Exploration Exercises. As we discuss this better way, keep the graphical interpretation of linear equations in mind.

Suppose there is an ordered pair (x, y) such that (3) and (4). If $5x - 3y - 11 = 0$, then for each k, $k(5x - 3y - 11) = 0$. [What principle for real numbers justifies this?] Similarly, if $7x - 6y - 10 = 0$, then for each m, $m(7x - 6y - 10) = 0$. But, if $k(5x - 3y - 11)$ is 0 and $m(7x - 6y - 10)$ is 0 then

$$k(5x - 3y - 11) + m(7x - 6y - 10) = 0. \qquad \text{[Why?]}$$

That is, then

$$(5) \qquad (5k + 7m)x + (-3k - 6m)y + (-11k - 10m) = 0.$$

Now, this last equation looks worse than the equations we started with! But, observe that (5) holds for all k and m. So, we can convert (5) to a simpler equation by picking values for 'k' and for 'm' in a clever way. A good choice would be one for which either '5k + 7m' or '−3k - 6m' has the value 0 [Why 0?]. For example, suppose we choose 7 for 'k' and −5 for 'm'. Then, the value of '5k + 7m' is 0, and (5) is converted into:

$$(6) \qquad (35 - 35)x + (-21 + 30)y + (-77 + 50) = 0,$$

which is equivalent to:

$$(7) \qquad 9y - 27 = 0$$

So, if there is an ordered pair (x, y) which satisfies both (3) and (4) then its second component is 3. Each of (3) and (4) tells us that its first component must be 4. So, if the system of (3) and (4) has a nonempty solution set, its only member is (4, 3). Since (4, 3) satisfies both (3) and (4) [as we discover by checking], the system has a nonempty solution set. So, its solution set is $\{(4, 3)\}$.

We could have selected values for 'k' and 'm' such that the value of '3k - 6m' is 0. If we take 2 for 'k' and −1 for 'm', (5) is converted into:

$$(8) \qquad (10 - 7)x + (-6 + 6)y + (-22 + 10) = 0,$$

which is equivalent to '3x - 12 = 0', and to 'x = 4'.

The foregoing is, of course, largely a discussion of a procedure for solving a system of linear equations in two variables. In practice, you might [with a different example] proceed as follows.

$$\begin{array}{ll} (1) & 2x + 5y - 31 = 0 \\ (2) & 3x - 7y + 26 = 0 \end{array}\Bigg\}$$

$$\begin{array}{rl} 6x + 15y - 93 = 0 & \text{[Multiply (1) by 3.]} \\ \underline{-6x + 14y - 52 = 0} & \text{[Multiply (2) by −2.]} \\ 29y - 145 = 0 & \text{[Add.]} \\ y = 5 & \end{array}$$

$$2x + 5 \cdot 5 - 31 = 0 \quad \text{[Substitute in (1).]}$$

$$2x - 6 = 0$$

$$x = 3$$

<u>Check</u>. $3 \cdot 3 - 7 \cdot 5 + 26 \stackrel{?}{=} 0$ [Substitute in (2).]

$$9 - 35 + 26 = 0 \checkmark$$

The solution set is $\{(3, 5)\}$.

The equations in the system need not be in standard form in order to apply this procedure. But, it does help if the x-terms and the y-terms are lined up. This enables you to tell at a glance what numbers to use as multipliers so that when you add, the resulting equation will contain but one variable. For example, consider the system:

$$\left.\begin{array}{ll} (1) & 9x - 7y = 16 \\ (2) & 4y = 3 - 3x \end{array}\right\}$$

Transform (2) to give the equivalent system:

$$\left.\begin{array}{ll} (1) & 9x - 7y = 16 \\ (2') & 3x + 4y = 3 \end{array}\right\}$$

Transform (2′) by multiplying by -3. Now, we have the system:

$$\left.\begin{array}{ll} (1) & 9x - 7y = 16 \\ (2'') & -9x - 12y = -9 \end{array}\right\}$$

Next, add:

$$-19y = 7,$$

$$y = -\frac{7}{19}$$

Substitute in (1):

$$9x - 7 \cdot -\frac{7}{19} = 16$$

$$9x + \frac{49}{19} = 16$$

$$171x + 49 = 304$$

$$171x = 255$$

$$x = \frac{255}{171} = \frac{85}{57}$$

Check in (2):

$$
\begin{array}{r||l}
\multicolumn{2}{c}{4\cdot -\frac{7}{19} \stackrel{?}{=} 3 - 3\cdot \frac{85}{57}} \\
-\frac{28}{19} & 3 - \frac{85}{19} \\
 & \frac{57}{19} - \frac{85}{19} \\
 & -\frac{28}{19} \checkmark
\end{array}
$$

Notice that we check in (2) rather than in (2''). Do you see why?

<u>Answer</u>. The solution set is $\{(\frac{85}{57}, -\frac{7}{19})\}$.

EXERCISES

<u>A</u>. Solve these systems of equations.

1. $\left.\begin{array}{l} 4x + 3y = -1 \\ 7x + 4y = -2 \end{array}\right\}$

2. $\left.\begin{array}{l} 6y + 2x = 34 \\ 5y + 3x = 35 \end{array}\right\}$

3. $\left.\begin{array}{r} 2r + 3s = -9 \\ r - 2s = 27 \end{array}\right\}$

4. $\left.\begin{array}{r} 10x - 7y = 0 \\ 3x + 4y = 0 \end{array}\right\}$

5. $\left.\begin{array}{l} x = 12 - 3y \\ x = y + 7 \end{array}\right\}$

6. $\left.\begin{array}{l} y = 5 - 2x \\ x = 3 - 4y \end{array}\right\}$

7. $\left.\begin{array}{l} 2x + 7y - 24 = 0 \\ 3x - 5y - 5 = 0 \end{array}\right\}$

8. $\left.\begin{array}{l} 7x + 4y - 17 = 0 \\ 3x - 2y - 11 = 0 \end{array}\right\}$

9. $\left.\begin{array}{l} 4x - 5y = 22 \\ 4 = y + 2x \end{array}\right\}$

10. $\left.\begin{array}{l} 6x + 7(2y + 3) = 0 \\ 5x = 6y - 44 \end{array}\right\}$

11. $\left.\begin{array}{l} x = 4y + 2 \\ 2x + 3y = 15 \end{array}\right\}$

12. $\left.\begin{array}{l} 3x - 5y = 5 \\ x = 9 - 2y \end{array}\right\}$

13. $\left.\begin{array}{l} y = 17 - 2x \\ x = y + 5 \end{array}\right\}$

14. $\left.\begin{array}{l} y = 3x - 2 \\ y = 7x + 5 \end{array}\right\}$

15. $\left.\begin{array}{l} 3y = 9x + 2 \\ 7x + 2y = 15 \end{array}\right\}$

16. $\left.\begin{array}{l} \frac{x - y}{3} = 5 \\ \frac{x + y}{5} = 3 \end{array}\right\}$

17. $\left.\begin{array}{l} x - 13y = -3 - x \\ 3(x + 2y) = 23 - 2y \end{array}\right\}$

18. $\left.\begin{array}{r} 1 + 2y + 3x = 0 \\ 9(x - 1) = 2 \end{array}\right\}$

19. $\left.\begin{array}{l} 5y + 7 = 2x \\ 2(3x - 5y) = 3(x - 4) - x \end{array}\right\}$

20. $\left.\begin{array}{l} 3x + 2(4y + 1) = 0 \\ 5(2x + 5y) - x = y - 6 \end{array}\right\}$

21. $\left.\begin{array}{r} 6(x - 3) + 7(2 - y) = 8 \\ 12(x + 4) - 3(6 - y) = 15 \end{array}\right\}$

22. $\left.\begin{array}{l} 2(x - 3y + 4) = 5(3x + y - 5) \\ x = 2y + 7 \end{array}\right\}$

23. $\left.\begin{array}{l} 3x - 2(x - y) = 15 \\ 5x - 5(x - y) = 70 \end{array}\right\}$

24. $\left.\begin{array}{l} 3(x - 2) - 2(3 - y) = 2y \\ 3(y - 2) - 2(3 - x) = 2x \end{array}\right\}$

B. Suppose that $\{(x, y)\colon ax + by + c = 0\}$ and $\{(x, y)\colon a'x + b'y + c' = 0\}$ are linear functions, f and g. [This means, of course, that $aba'b' \neq 0$.]

1. Prove that f and g have the same slope if and only if a, b, a′, and b′ are in proportion.
2. What can you say about f and g if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$?
3. What can you say about f and g if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$?

INDEPENDENT, DEPENDENT, AND INCONSISTENT EQUATIONS

Your work in Part B may have suggested that a system of two linear equations in two variables 'x' and 'y':

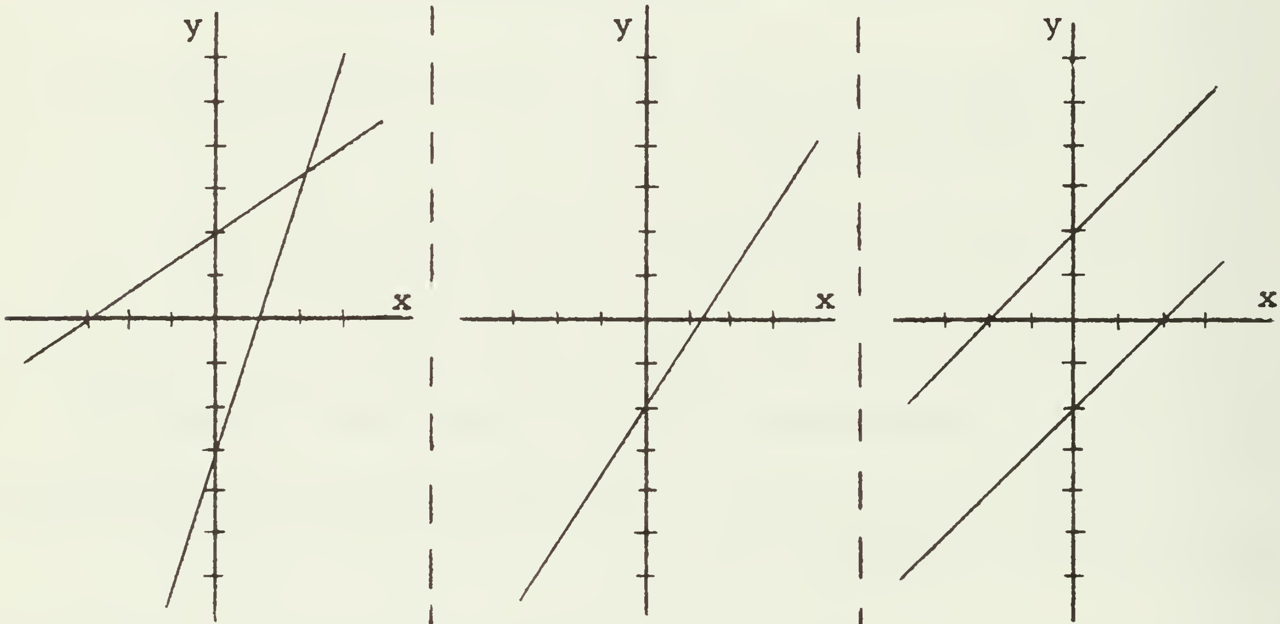
$$\left.\begin{array}{l} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{array}\right\} aba'b' \neq 0$$

has precisely one solution if and only if $ab' \neq a'b$. If $ab' \neq a'b$, it follows that $\frac{a}{b} \neq \frac{a'}{b'}$, and $-\frac{a}{b} \neq -\frac{a'}{b'}$. But, $-\frac{a}{b}$ and $-\frac{a'}{b'}$ are the slopes of the linear functions corresponding to the given equations. So, if $ab' \neq a'b$, the corresponding linear functions have different slopes. Hence, their intersection consists of just one point. Now, if the slopes are the same, either the functions are the same or the functions intersect in the empty set. So, if the slopes are the same, the intersection does not consist of precisely one point. Therefore, the intersection consists of just one point if and only if the slopes are different. And, from this it follows that the intersection consists of just one point if and only if $ab' \neq a'b$.

A system of equations which has at least one solution is called *a consistent system of equations*; one which has no solution is called *an inconsistent system of equations.*

A consistent system of two linear equations in two variables either has precisely one solution or consists of two equations which have the same graph. In the first case [precisely one solution], the equations are said to be *independent equations*; in the second case [same graph], they are said to be *dependent equations*.

Two linear equations in two variables are independent [and, so, have precisely one solution] if and only if $ab' \neq a'b$. So, if $ab' = a'b$, the equations are either dependent or inconsistent. [Equations such that $aba'b' \neq 0$ are dependent if and only if $ab' = a'b$ and $bc' = b'c$; they are inconsistent if and only if $ab' = a'b$ and $bc' \neq b'c$.]



Independent Equations	Dependent Equations	Inconsistent Equations
$ab' \neq a'b$	$ab' = a'b,\ bc' = b'c$	$ab' = a'b,\ bc' \neq b'c$

EXERCISES

A. For each system, tell whether it is a system of independent linear equations, a system of dependent linear equations, or a system of inconsistent linear equations. If you claim that the equations are independent, find the ordered pair which is a solution.

Sample. $\left.\begin{array}{l} 2(x - 4y) + 1 = 3(x + 3y) - (2x - y) + 6 \\ 7(x - 2y + 7) = 4(x - 5y + 6) \end{array}\right\}$

Solution. First, we transform these equations into standard linear form.

$$(1) \qquad x - 18y - 5 = 0$$

$$(2) \qquad 3x + 6y + 25 = 0$$

Since $1 \cdot 6 \neq 3 \cdot -18$, or, equivalently, since $\frac{1}{3} \neq \frac{-18}{6}$, the equations are independent. So, there is one and only one ordered pair (x, y) which satisfies them. Let's find it. From equation (1) we know that

$$x = 18y + 5.$$

Substituting '18y + 5' for 'x' in (2), we get:

$$3(18y + 5) + 6y + 25 = 0,$$

$$54y + 15 + 6y + 25 = 0,$$

$$60y = -40,$$

$$y = -\frac{2}{3}$$

$$x = 18 \cdot -\frac{2}{3} + 5 = -7$$

Check. $$3 \cdot -7 + 6 \cdot -\frac{2}{3} + 25 \stackrel{?}{=} 0$$

$$-21 + -4 + 25 = 0 \quad \checkmark$$

The unique solution is $(-7, -\frac{2}{3})$.

1. $\left.\begin{matrix} 5x + 3y - 12 = 0 \\ 6x - 7y - 4 = 0 \end{matrix}\right\}$

2. $\left.\begin{matrix} 9x = 12 - 7y \\ 3x = 4 - 2y \end{matrix}\right\}$

3. $\left.\begin{matrix} 8t = 3s + 7 \\ 6s = 16t - 15 \end{matrix}\right\}$

4. $\left.\begin{matrix} x + y = 0 \\ 3x - 2y = 0 \end{matrix}\right\}$

5. $\left.\begin{matrix} a + 8b = 3 \\ 8a + b = 3 \end{matrix}\right\}$

6. $\left.\begin{matrix} 7t + 3a = -6 \\ -7t - 3a = 12 \end{matrix}\right\}$

7. $\left.\begin{matrix} 4x - 2y = 9 \\ y = 2x + 4 \end{matrix}\right\}$

8. $\left.\begin{matrix} 8k - 7 = m \\ 8k + 7 = m \end{matrix}\right\}$

9. $\left.\begin{matrix} 2y + x = 5 - y \\ 4y - x = y - 5 \end{matrix}\right\}$

10. $\left.\begin{matrix} 6(x + z) - z = 3(x + 1) + 1 \\ 4(2x + 4z - 3) = z - x \end{matrix}\right\}$

11. $\left.\begin{aligned} 2x - 3y &= 5 \\ 4x - 6y &= 11 \end{aligned}\right\}$

12. $\left.\begin{aligned} 2x - 3y &= 5 \\ 4y - 6x &= 11 \end{aligned}\right\}$

13. $\left.\begin{aligned} \frac{35}{12}A - \frac{5}{4}B &= 1 \\ \frac{14}{15}A - \frac{2}{5}B &= 2 \end{aligned}\right\}$

14. $\left.\begin{aligned} y &= \frac{6 - 2x}{7} \\ x &= \frac{9 + 10.5y}{3} \end{aligned}\right\}$

15. $\left.\begin{aligned} &r + 5s - 6(r - 3s) + 7(x - r) = 3s \\ &3(2r - 7s) + 5(r + 2s + 3) = 6(1 - r) \end{aligned}\right\}$

16. $\left.\begin{aligned} &7a + 3(a - 3b + 1) = 1 + 2(2a + b + 8) - 3b \\ &7(2a - 3b - 5) + a(2b + 1) = b(2a - 1) \end{aligned}\right\}$

17. $\left.\begin{aligned} &(u - 5)^2 - (v + 4)^2 + 10u = 12 + (u - v)(u + v) \\ &(v - 8)^2 - (u + v)^2 = -2uv - (u + 1)^2 \end{aligned}\right\}$

☆ B. Solve this system for 'x' and 'y'. [That is, express 'x' and 'y' in terms of 'a', 'b', 'c', 'a′', 'b′', and 'c′'.]

$$\left.\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}\right\}$$

How does the result show the cases in which the equations are inconsistent or dependent?

SYSTEMS OF THREE EQUATIONS IN THREE VARIABLES

The ideas involved in solving systems of two linear equations in two variables can be extended to solving systems of three linear equations in three variables. In general, the procedure followed in solving a system of two linear equations in two variables is to obtain a single equation in one variable by <u>eliminating the other variable</u> from the given equations. You already know how to solve the resulting equation in one variable, and you have seen how doing so enables you to solve the given system.

When you are faced with the problem of solving a system of three linear equations in three variables, say:

$$\left.\begin{aligned} 5x + 2y + 3z - 8 &= 0 \\ 3x - 4y - z + 14 &= 0 \\ 2x + 7y - 3z - 3 &= 0 \end{aligned}\right\},$$

a natural procedure is to try to "reduce" this problem to one of solving a system of two linear equations in two variables [just as you "reduced" the problem of solving a system of two linear equations in two variables to one of solving one equation in one variable]. This can be done by choosing one of the variables and eliminating it, first from one pair of the given equations, and then from another pair. For example, if you want to eliminate 'z', take the first two equations of the system:

$$5x + 2y + 3z - 8 = 0,$$
$$3x - 4y - z + 14 = 0$$

Then, transform them and add, thus eliminating 'z'.

$$\begin{array}{lrl} & 5x + 2y + 3z - 8 &= 0 \\ & 9x - 12y - 3z + 42 &= 0 \\ \hline \text{(a)} & 14x - 10y \qquad\quad + 34 &= 0 \end{array}$$

Then, eliminate 'z' from, say, the first and third equations of the given system.

$$\begin{array}{lrl} & 5x + 2y + 3z - 8 &= 0 \\ & 2x + 7y - 3z - 3 &= 0 \\ \hline \text{(b)} & 7x + 9y \qquad\quad - 11 &= 0 \end{array}$$

Equations (a) and (b) make a system of two linear equations in two variables.

$$\left.\begin{array}{lr} \text{(a)} & 14x - 10y + 34 = 0 \\ \text{(b)} & 7x + 9y - 11 = 0 \end{array}\right\}$$

If x, y, and z satisfy the equations in the given system, x and y must satisfy the two equations in this last system [Why?]. We solve this system of two equations in two variables and find that the solution set is $\{(-1, 2)\}$.

Substitute in, say, the first of the three equations in the given system:

$$5 \cdot -1 + 2 \cdot 2 + 3z - 8 = 0,$$
$$z = 3$$

So, $(-1, 2, 3)$ is a solution of the first equation. Is it a solution of both the second and third equations? Is it the only solution of the given system?

Another procedure for eliminating variables is by substitution. Reconsider the system dealt with on the preceding page:

$$\left.\begin{array}{ll}(1) & 5x + 2y + 3z - 8 = 0 \\ (2) & 3x - 4y - z + 14 = 0 \\ (3) & 2x + 7y - 3z - 3 = 0\end{array}\right\}$$

Solve (2) for 'z': $z = 3x - 4y + 14$

Then, substitute in (1) and in (3):

$$\left.\begin{array}{ll}(1') & 5x + 2y + 3(3x - 4y + 14) - 8 = 0 \\ (2') & 2x + 7y - 3(3x - 4y + 14) - 3 = 0\end{array}\right\}$$

This last system is one of two equations in two variables. Complete the solution.

EXERCISES

Solve these systems.

1. $\left.\begin{array}{l}2x - 3y + 5z - 11 = 0 \\ 7x + y - 3z = 0 \\ 3x - 2y + 7z = 20\end{array}\right\}$

2. $\left.\begin{array}{l}5x = 12 - y + z \\ 2y = x - z + 3 \\ z = x + y\end{array}\right\}$

3. $\left.\begin{array}{l}x + y + z = 1 \\ 2x + z = 13 \\ z - y = 10\end{array}\right\}$

4. $\left.\begin{array}{l}x + 2y + 3z = 4 \\ 7 - z = 2y + 3x \\ 3y + 4z = 1 - 2x\end{array}\right\}$

5. $\left.\begin{array}{l}4(3x - y) = 3(17 - 3z) \\ x + y - 1 = 1 - 6z \\ 2(x - 10) = 7(y + 3z)\end{array}\right\}$

6. $\left.\begin{array}{l}x + y = 5 - 2z \\ x + 2y = 9 - z \\ y + 2x = 15 - z\end{array}\right\}$

7. $\left.\begin{array}{l}x + y = w - z \\ 3x + y - z = 2w \\ x = y - w \\ w + 2z = x - 1\end{array}\right\}$

8. $\left.\begin{array}{l}x + y + u = 4 \\ y + u + v = -5 \\ u + v + x = 0 \\ v + x + y = -8\end{array}\right\}$

☆ 9. $\left.\begin{array}{l}4x + 1 + 3z + 8y = 0 \\ z + 10y + 4 + 5x = 0 \\ 14y + 5z + 7x + 2 = 0\end{array}\right\}$

☆ 10. $\left.\begin{array}{l}x + y + 4z = 7 \\ 2x + 2y + 8z = 14 \\ 3x + 3y + 12z = 21\end{array}\right\}$

SYSTEMS OF NONLINEAR EQUATIONS "IN LINEAR FORM"

Procedures used in handling systems of linear equations can be applied to systems of nonlinear equations which are linear in form.

Example 1.

$$\left.\begin{aligned} 2\left(\frac{1}{x}\right) - 3\left(\frac{1}{y}\right) - 17 &= 0 \\ 5\left(\frac{1}{x}\right) + 6\left(\frac{1}{y}\right) - 2 &= 0 \end{aligned}\right\}$$

[These equations are linear in '$\frac{1}{x}$' and '$\frac{1}{y}$'.]

Solution.

$$\begin{aligned} 4\left(\frac{1}{x}\right) - 6\left(\frac{1}{y}\right) - 34 &= 0 \\ 5\left(\frac{1}{x}\right) + 6\left(\frac{1}{y}\right) - 2 &= 0 \\ \hline 9\left(\frac{1}{x}\right) \qquad\qquad - 36 &= 0 \\ \frac{1}{x} &= 4 \\ x &= \frac{1}{4} \\ 2(4) - 3\left(\frac{1}{y}\right) - 17 &= 0 \\ -3\left(\frac{1}{y}\right) &= 9 \\ \frac{1}{y} &= -3 \\ y &= -\frac{1}{3} \end{aligned}$$

Check.

$$5\left(\frac{1}{\frac{1}{4}}\right) + 6\left(\frac{1}{-\frac{1}{3}}\right) - 2 \stackrel{?}{=} 0$$

$$20 + -18 - 2 = 0$$

The solution set is $\{(\frac{1}{4}, -\frac{1}{3})\}$.

Example 2.

$$\left.\begin{aligned} 3x^2 - 2y^2 + 6 &= 0 \\ 2x^2 + 5y^2 - 53 &= 0 \end{aligned}\right\}$$

[These equations are linear in 'x^2' and 'y^2'.]

Solution.

$$\begin{aligned} 6x^2 - 4y^2 + 12 &= 0 \\ 6x^2 + 15y^2 - 159 &= 0 \\ \hline -19y^2 + 171 &= 0 \\ y^2 &= 9 \end{aligned}$$

$$y = 3 \text{ or } y = -3$$

$$3x^2 - 2 \cdot 9 + 6 = 0$$

$$x^2 = 4$$

$$x = 2 \text{ or } x = -2$$

Check. $$2 \cdot 4 + 5 \cdot 9 - 53 \stackrel{?}{=} 0$$

$$8 + 45 - 53 = 0 \quad \checkmark$$

The solution set is $\{(2, 3), (2, -3), (-2, 3), (-2, -3)\}$.

EXERCISES

A. Solve these systems.

1. $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 10 \\ \frac{1}{x} - \frac{1}{y} &= 3 \end{aligned}\right\}$

2. $\left.\begin{aligned} \frac{3}{x} - \frac{5}{y} &= 6 \\ \frac{4}{x} + \frac{3}{y} &= 37 \end{aligned}\right\}$

3. $\left.\begin{aligned} x + 2y &= 6 \\ \frac{x}{4} + \frac{y}{5} &= 2 \end{aligned}\right\}$

4. $\left.\begin{aligned} 3x^2 - \sqrt{y} &= 9 \\ 2x^2 + \sqrt{y} &= 11 \end{aligned}\right\}$

5. $\left.\begin{aligned} 6x - \frac{13}{y} &= 2 \\ 5x - \frac{12}{y} &= 4 \end{aligned}\right\}$

6. $\left.\begin{aligned} 2x^2 &= y^2 + 14 \\ 2y^2 &= x^2 + 47 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3x^2 - 5y^2 &= 4 \\ y^2 - 4 &= 0 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x^2 + 2y &= 27 \\ x^2 - y &= 4 \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{1}{2x} + \frac{2}{3y} &= 18 \\ \frac{3}{4x} + \frac{4}{5y} &= 21 \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{x+y}{2} - \frac{x-y}{3} &= 8 \\ \frac{x+y}{3} + \frac{x-y}{4} &= 11 \end{aligned}\right\}$

☆ 11. $\left.\begin{aligned} x + y^2 + \frac{1}{z} &= 8 \\ 3x + y^2 - \frac{1}{z} &= 4 \\ 2x - 3y^2 + \frac{3}{z} &= -1 \end{aligned}\right\}$

☆ 12. $\left.\begin{aligned} \frac{1}{x} &= \frac{1}{z} - \frac{2}{y} \\ 2\left(\frac{1}{y} - \frac{1}{x}\right) &= \frac{1}{z} \\ \frac{3}{y} - \frac{5}{x} - 1 &= \frac{1}{z} \end{aligned}\right\}$

B. 1. Find an equation whose graph is a straight line which contains the graphs of $(7, 3)$ and $(-6, 2)$.

2. Find an equation whose graph is a parabola with a vertical axis of symmetry and which contains the points (4, 51), (−3, 16), and (1, 0).

☆ 3. Find an equation whose graph is a parabola with a horizontal axis of symmetry and which contains the points (1, 1), (−25, 3), and (2, 0).

☆ 4. $\{(x, y): y = 3x^2 - 5x + 1\} \cap \{(x, y): y = 13x - 23\} = ?$

USING SYSTEMS TO SOLVE PROBLEMS

Many of the problems you solved in earlier units by using one equation in one variable can be solved by using a system of two equations in two variables. Frequently, it is easier to "set up" a system of equations than it is to set up a single equation. Of course, you should get the correct answer to the problem in either case.

Example 1. A cashier has 8 more five-dollar bills than she has one-dollar bills. The total value of these bills is $202. How many bills of each denomination does she have?

Solution. Suppose the cashier has w one-dollar bills and f five-dollar bills. Then, $f = w + 8$. Moreover, since the five-dollar bills are worth 5f dollars and the one-dollar bills are worth w dollars, the total value of these bills is $5f + w$ dollars. So, $5f + w = 202$. Hence, we are looking for numbers, f and w, such that

$$\left.\begin{array}{lll} & (1) & f = w + 8 \\ \text{and} & (2) & 5f + w = 202. \end{array}\right\}$$

We solve this system by substitution.

$$5(w + 8) + w = 202$$

$$5w + 40 + w = 202$$

$$6w = 202$$

$$w = 27$$

$$f = 27 + 8 = 35$$

So, the cashier has 27 one-dollar bills and 35 five-dollar bills.

Check. 35 is 8 more than 27. The 35 five-dollar bills are worth 175 dollars. $175 + 27 = 202$.

Example 2. A grocer wishes to blend two grades of coffee to obtain a mixture of 100 pounds which he can sell for 89 cents per pound. He has 85-cent coffee and 95-cent coffee. How much of each grade of coffee should he use in the mixture?

Solution. Suppose he mixes c pounds of 85-cent coffee with n pounds of 95-cent coffee. Then, $c + n = 100$. Also, this mixture is worth $85c + 95n$ cents. So, $85c + 95n = 89 \cdot 100$. Hence, we are looking for numbers, c and n, such that

$$\left.\begin{array}{lll} & (1) & c + n = 100 \\ \text{and} & (2) & 85c + 95n = 8900. \end{array}\right\}$$

Solve this system and check.

Example 3. I am thinking of an integer between 9 and 100. I interchange the digits in its decimal numeral. This gives me a number which is 3 less than 4 times the original number. Find the original number if the digits stand for numbers whose sum is 7.

Solution. The decimal numeral contains two digits [Why?]. Suppose the tens digit is t and the units digit is u. Then, the original number is $10t + u$. The new number is $10u + t$. So, $10u + t = 4(10t + u) - 3$. Also, $t + u = 7$. Hence, we want numbers t and u such that

$$\left.\begin{array}{lll} & (1) & 10u + t = 4(10t + u) - 3 \\ \text{and} & (2) & t + u = 7. \end{array}\right\}$$

We solve equation (2) for 't':

$$t = 7 - u,$$

and then substitute for 't' in equation (1).

$$
\begin{aligned}
10u + (7 - u) &= 4[10(7 - u) + u] - 3 \\
10u + 7 - u &= 4[70 - 10u + u] - 3 \\
10u + 7 - u &= 280 - 40u + 4u - 3 \\
9u + 7 &= 277 - 36u \\
45u &= 270 \\
u &= 6 \\
t &= 7 - 6 = 1
\end{aligned}
$$

The original number is 16.

<u>Check.</u> 61 is $4 \cdot 16 - 3$, and $1 + 6$ is 7.

EXERCISES

1. A jar of dimes and nickels contains 120 coins worth a total of \$7.50. How many dimes are there in the jar? How many nickels?

2. Pat found an envelope containing 104 stamps. Some were 3-cent stamps and the rest 2-cent stamps. The stamps were worth \$2.75. How many 3-cent and how many 2-cent stamps did Pat find.

3. An importer blends 20 pounds of tea by mixing 47-cent tea and 52-cent tea. The mixture is to be sold at 50 cents a pound. How many pounds of each kind of tea must be mixed?

4. A housewife decides to save money by mixing powdered milk and whole milk. Whole milk costs 24 cents a quart and skim milk made from powder will cost 6 cents a quart. If the family uses 90 quarts of milk per month and if the housewife decided to spend 5 cents per quart, how many quarts of whole milk must she use each month?

5. Two boys run a lemonade stand. They sold 850 cups of lemonade on a certain day, some at 5 cents per cup, the rest at 10 cents per cup. Their total receipts were \$55. How many cups were sold at each price?

6. Find two numbers whose difference is 20 and such that seven percent of the smaller added to five percent of the larger gives the sum 4.

7. The individual digits in a two-digit decimal numeral stand for numbers whose sum is 6. When the digits are interchanged, the new numeral stands for a number which is 1/10 of the original number. Find the original number.

8. A man invests a total of $25,000 in two enterprises. He invests a part of this money at 3% and the remainder at 4%. How much does he invest at each rate if his annual income is $950?

9. Mr. Adams invested $5,000 in two enterprises, one bearing interest at 5.5% and the other at 6.5%. If his total annual income from these investments is $305, how much did he invest in each enterprise?

10. Eight times the sum of the numbers named by the individual digits in a two-digit decimal numeral is equal to the number named by the two-digit numeral. Interchanging the digits of the numeral gives the name of a new number which is 1 less than 4 times the number named by the tens digit of the given numeral. Find the original number.

11. Mrs. Charles invested $500 more at 3% than she invested at 4%. If she received the same income from both investments, how much did she invest at each rate?

12. The sum of two numbers is 25. One number is 2 more than twice the other. Find the two numbers.

13. The average of two numbers is 13. The larger is 2 more than the smaller. Find the numbers.

14. The foot-perimeter of a rectangular lot is 360. Its length is 20 feet more than the width. What are its dimensions?

15. A bottle and a cork together cost $1.10. If the bottle costs $1.00 more than the cork, what is the cost of each?

16. Find an ordered pair of numbers whose sum if 58 and whose difference, second from first, is 114.

17. The sum of three times one number and twice the second number is 50. Four percent of the second number added to six percent of the first number gives the sum 1. Find the two numbers.

18. The difference between two numbers is 5. The larger number is 4 more than 3 times the smaller. What are the numbers?

19. Thirty-seven coins consisting of nickels and dimes are worth \$2.30. How many coins of each kind are there?

20. The difference between two numbers is 5. If the larger number is 5 times the smaller, what are the numbers?

21. Bill's present age in years is 1 more than six times Andy's age in years. If both boys were twice as old as they actually are, then Andy would be 3 years younger than Bill. How old is each boy now?

22. Milton bought 2 red apples and 4 green ones for 19 cents. Oswald bought 4 of the red apples and 2 green ones for 23 cents. What was the price of each kind?

23. Mr. Jones invested a total of \$23,000 in two businesses. He receives a 3% return from one business and a 4.5% return from the other. His annual income from these two investments is \$817.50. How much did he invest at each rate?

24. The sum of two numbers is 1200. Three times the smaller plus twice the larger equals 2704. What are the numbers?

25. A box contains 14 jars. Some are pint jars, the rest are quart jars. The 14 jars have a combined capacity of 19 pints. How many jars of each size are there in the box?

26. How many pounds of 32-cent rice and how many pounds of 27-cent rice must be mixed to make 50 pounds of 24-cent rice?

27. The width of a rectangle is 2/3 of its length and its perimeter is 80. Find its dimensions.

28. Three years ago Mary was three times as old as John. Three years hence she will be four times as old as John will be. Find the present age of each.

29. A man drew \$75.00 in one-dollar bills and five-dollar bills from the bank. There were 3 more one-dollar bills than five-dollar bills. How many bills of each kind were threr?

30. Ruth and Marilyn can do a typing job together in 4 hours. If Ruth works alone on the job for 3 hours, Marilyn can finish the job by herself in 6 hours. How long would it take each typist to do the entire job if she worked alone?

31. If 3 is added to the numerator-number and to the denominator-number of a fraction, the ratio of the first sum to the second is $\frac{6}{7}$. If 2 is subtracted from both numerator-number and denominator-number of the original fraction, the ratio of the first difference to the second is $\frac{1}{2}$. What is the original fraction?

32. The individual digits in a three-digit decimal numeral stand for numbers whose sum is 3 more than 3 times the number named by the hundreds digit. If the tens digit is interchanged with the hundreds digit, the new numeral stands for a number which is 7.3 times the difference of the original number from the new number. In the original numeral, the units digit names a number which is two more than the number named by the hundreds digit. What is the original numeral?

33. The area-measure of the cylindrical surface of a gasoline storage tank is 500π, and its height-measure is $\frac{4}{5}$ of its diameter-measure. What should be the height of a similar tank which has the same area-measure but is 4 feet smaller in diameter? [Hint. A formula for finding the area-measure of a cylindrical surface is '$S = 2\pi rh$'.]

[Supplementary exercises are in Part W, pages 5-275 through 5-278.]

TABLE OF SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1.000	3.162	51	2601	7.141	22.583
2	4	1.414	4.472	52	2704	7.211	22.804
3	9	1.732	5.477	53	2809	7.280	23.022
4	16	2.000	6.325	54	2916	7.348	23.238
5	25	2.236	7.071	55	3025	7.416	23.452
6	36	2.449	7.746	56	3136	7.483	23.664
7	49	2.646	8.367	57	3249	7.550	23.875
8	64	2.828	8.944	58	3364	7.616	24.083
9	81	3.000	9.487	59	3481	7.681	24.290
10	100	3.162	10.000	60	3600	7.746	24.495
11	121	3.317	10.488	61	3721	7.810	24.698
12	144	3.464	10.954	62	3844	7.874	24.900
13	169	3.606	11.402	63	3969	7.937	25.100
14	196	3.742	11.832	64	4096	8.000	25.298
15	225	3.873	12.247	65	4225	8.062	25.495
16	256	4.000	12.649	66	4356	8.124	25.690
17	289	4.123	13.038	67	4489	8.185	25.884
18	324	4.243	13.416	68	4624	8.246	26.077
19	361	4.359	13.784	69	4761	8.307	26.268
20	400	4.472	14.142	70	4900	8.367	26.458
21	441	4.583	14.491	71	5041	8.426	26.646
22	484	4.690	14.832	72	5184	8.485	26.833
23	529	4.796	15.166	73	5329	8.544	27.019
24	576	4.899	15.492	74	5476	8.602	27.203
25	625	5.000	15.811	75	5625	8.660	27.386
26	676	5.099	16.125	76	5776	8.718	27.568
27	729	5.196	16.432	77	5929	8.775	27.749
28	784	5.292	16.733	78	6084	8.832	27.928
29	841	5.385	17.029	79	6241	8.888	28.107
30	900	5.477	17.321	80	6400	8.944	28.284
31	961	5.568	17.607	81	6561	9.000	28.460
32	1024	5.657	17.889	82	6724	9.055	28.636
33	1089	5.745	18.166	83	6889	9.110	28.810
34	1156	5.831	18.439	84	7056	9.165	28.983
35	1225	5.916	18.708	85	7225	9.220	29.155
36	1296	6.000	18.974	86	7396	9.274	29.326
37	1369	6.083	19.235	87	7569	9.327	29.496
38	1444	6.164	19.494	88	7744	9.381	29.665
39	1521	6.245	19.748	89	7921	9.434	29.833
40	1600	6.325	20.000	90	8100	9.487	30.000
41	1681	6.403	20.248	91	8281	9.539	30.166
42	1764	6.481	20.494	92	8464	9.592	30.332
43	1849	6.557	20.736	93	8649	9.644	30.496
44	1936	6.633	20.976	94	8836	9.695	30.659
45	2025	6.708	21.213	95	9025	9.747	30.822
46	2116	6.782	21.448	96	9216	9.798	30.984
47	2209	6.856	21.679	97	9409	9.849	31.145
48	2304	6.928	21.909	98	9604	9.899	31.305
49	2401	7.000	22.136	99	9801	9.950	31.464
50	2500	7.071	22.361	100	10000	10.000	31.623

MISCELLANEOUS EXERCISES

A. Simplify.

1. $(2x + 3)(2x - 3)$
2. $(3x + 1) + (4x - 3) + (x + 2)$
3. $5a + 4b - (2a - 3b)$
4. $12x^4y^3 \div (4xy^3)$
5. $\frac{2x + 3}{3} + \frac{x - 2}{2}$
6. $\frac{2x + 3}{3x} + \frac{x - 2}{2x}$
7. $\sqrt{2} + 3\sqrt{2} - 11\sqrt{2}$
8. $7k - 3m - (5k - 3m)$
9. $12y^3z^2 \div (6yz^2)$
10. $(3x + 5)(x - 4)$
11. $4m - 3n - (3m + 2n)$
12. $7a + 5b + 9 + b$
13. $\frac{5x}{3} + \frac{x}{4}$
14. $\frac{x^2 - 9}{5} \div \frac{2(x + 3)}{10}$
15. $24x^6y^2 \div (6x^2y)$
16. $2b + 3 + (5b - 6) + (b + 4)$
17. $(4a + 1)(4 + a)$
18. $5\sqrt{18} + \sqrt{2}$
19. $(2k - 5)(3k - 2)$
20. $3y + 4 + (5y - 3) + 2y - 1$
21. $28x^5y \div (14xy)$
22. $(7s - t)(7s + t)$
23. $15(x+1)^3y^4 \div [5(x+1)^3y^4]$
24. $(3x + 7y) + (5x - 4y) - (7x + 5y)$
25. $\sqrt{27} + \sqrt{3}$
26. $x^2 + 4x - 1 - (2x^2 - 3x + 2)$
27. $\frac{(x + 5)^2}{25} \cdot \frac{5}{x + 5}$
28. $\frac{x + y}{2} + \frac{y - x}{3}$
29. $(x + 2)^2 - 2(x - 2)$
30. $(2 - m)^2 - 3(m - 2)$
31. $(4 + x)(4 - x)$
32. $(4 + \sqrt{3})(4 - \sqrt{3})$

B. Factor.

1. $3km + 15m$ [Answer. $3m(k + 5)$]
2. $5k^4m + 70k^6m$ [Answer. $5k^4m(1 + 14k^2)$]
3. $4rx + 12r$
4. $7x^5y + 21x^3y$
5. $c^2 - 25$
6. $a^2 - 36$
7. $5a^3b^2 + 15a^2b^2$
8. $4a^5b^3 + 8a^3b^3$
9. $9x^2 - 1$
10. $x^2 - 49$
11. $3r^2s^3 + 6rs^2$
12. $4ab^2 + 20ab$

C. Solve.

1. $3m = 15$
2. $5r - 3 = 17$
3. $2x^2 = 18$
4. $7x + 3 = 5x - 9$
5. $5(t + 4) = 25$
6. $\frac{t}{3} + \frac{t}{4} = 7$
7. $\frac{3}{5} = \frac{b}{20}$
8. $4t = 20$
9. $3x + 2 = 14$
10. $\frac{4}{6} = \frac{x}{15}$
11. $\frac{x}{2} + \frac{2x}{3} = 7$
12. $4(s - 3) = 12$
13. $5x - 3 = 32$
14. $4p + 7 = 19$
15. $5x - 4 = 3x + 6$
16. $6(y+3) = 2(1+3y)$
17. $\frac{m}{7} = \frac{6}{3}$
18. $\frac{x}{3} + \frac{x}{4} = \frac{7}{12}$
19. $6b - 15 = 0$
20. $x^2 - 2x - 15 = 0$ and $x > 0$
21. $6 - 3x = 9$
22. $6y + 13 = 4y + 15$
23. $\frac{m}{2} + \frac{3m}{4} = 5$
24. $\frac{3}{x} = \frac{4}{6}$
25. $2m - 5 = 7$
26. $4(x + 6) + x = 8x - 3$
27. $4 = \frac{12}{b}$
28. $3x + 17 > 5 + 7x$
29. $2(3x - 1) < 5x + 4$
30. $\frac{r}{2} + \frac{r}{3} = 5$
31. $\frac{y+2}{3} - \frac{y-2}{5} = 2$
32. $3x + 1 < 8x + 16$
33. $\frac{8}{20} < \frac{x}{30}$
34. $10y - 2(3y + 1) > 26$

D. 1. If k feet of lumber cost m dollars, what is the cost of one foot?

2. For each x, $(x - 3)(\_\_\_\_\_\_) = x^2 - 9$.

3. In a class of 30 students, 6 were absent. What percent of the students were present?

4. The average distance between the sun and the Earth is 92,900,000 miles. If light travels at the rate of 186,300 miles per second, it takes __________ seconds for light from the sun to reach the Earth.

(a) less than 10
(b) between 10 and 100
(c) between 100 and 1000

5. Draw the straight line which contains the graphs of (3, 4) and (−6, −2). This line crosses the graph of the y-axis in the graph of what ordered pair?

6. For each x, $x^2 + 4x - 7 = (x + 2)(x + 2) +$ ________.

7. For each $x \neq 3$, $\dfrac{x^2 - 5x + 7}{x - 3} = x - 2 +$ ________.

E. Expand.

1. $(x + 4)(x - 9)$ [Answer. $x^2 - 5x - 36$]
2. $(x - 8)(x - 2)$
3. $(x - 7)(x + 8)$
4. $(x + 4)^2$
5. $(x - 5)^2$
6. $(3x - 1)^2$
7. $(5 - 2y)^2$
8. $(3 - x)(x + 5)$
9. $(2x + 7)(3x - 4)$
10. $(a^2 + 5)(a^2 - 5)$
11. $(m - n)(m^2 + n^2)(m + n)$
12. $3(x - 4)(x + 17)$
13. $\frac{1}{2}(2x - 5)(4x + 6)$
14. $(2Q - A)(2Q - A)$
15. $(A - 2Q)(A - 2Q)$

F. Factor.

1. $x^2 + 16x + 63$
2. $mm - 9m - 22$
3. $x^2 - 361$
4. $18 + p^2 - 19p$
5. $k^2 + 70k - 144$
6. $81 - y^4$
7. $D^2 + 2DE + E^2$
8. $2x^2 + 16x + 30$
9. $7y^2 - 7y - 42$
10. $5A^2 - 55A + 120$
11. $6F^2 - 13NF + 6N^2$
12. $d^2 + 6d - 91$

G. Solve.

1. $x(x - 7) = 0$
2. $c^2 - 30c + 200 = 0$
3. $2H^2 + 28H - 2 = 100$
4. $x^2 - 5x + 6 > 0$
5. $m^8 + 1728 = 1728 + m^8$
6. $5y^2 + 1 = y(2y + 3y)$

H. True or false?

1. $\forall_m \ |-3m| = 3m$
2. $\forall_x \ \sqrt{36x^2} = 6x$
3. $\forall_a \ 4a^2 \geq 2a$
4. $\forall_n \ \sqrt{n} \cdot \sqrt{n} = n$
5. '$x = 5$ and $x = 7$' is equivalent to '$x^2 - 12x + 35 = 0$'
6. $\sqrt{2} = 1.414$
7. $\forall_x$ $2x$ is an even number
8. $\exists_x \ 5(x - 1) = 3x + 2(7 + x)$
9. $\forall_x \ 5(3 - x) = 3(5 - x) - 2x$

I. 1. On a scale drawing of a house, $\frac{1}{8}$ inch on the drawing represents 1 foot. So, a 14-foot long room is represented by a line segment ________ inches long.

2. If n is an odd number, the next larger odd number is ________.

3. If $a = 3$ and $b = -2$ then $a - b =$ ________.

4. If the average gasoline consumption of a certain car is 15 miles to the gallon, how many gallons are needed to drive this car x miles?

5. Solve for 'x': $3x + 2 = y$

6. What percent of 80 is 60?

7. What is the average of $x - 8$ and $3x + 2$?

8. Reduce to lowest terms: $\dfrac{x^2 - 4y^2}{2x + 4y}$

9. Which is larger, $5\sqrt{3}$ or $3\sqrt{5}$?

J. True or false?

1. For each set B, $B \cup B = B$.
2. $\forall_B \ B \cup \emptyset = B$
3. $\forall_B \ B \cap \emptyset = \emptyset$
4. $\forall_B \ B \cap B = B$
5. $\forall_H \forall_K$ if K contains 6 elements $[n(K) = 6]$ and H contains 3 elements $[n(H) = 3]$, then $H \times K$ contains 18 elements $[n(H \times K) = 18]$.
6. $\forall_A \forall_B$ if $n(A \times B) = 81$ then $n(A) = n(B)$
7. $\forall_A \forall_B$ if $n(A \times B) = 49$ then $n(A) = n(B)$

8. $\forall_A \forall_B$ if $A = B$ then $A \times B = B \times A$.

9. $\forall_A \forall_B$ if $A \times B = B \times A$ then $A = B$.

10. Unioning is distributive over unioning.

K. 1. If a room has x rows of desks with 7 desks in a row, how many desks are there in the classroom?

2. True or false: $5 \in \{x: 2x + 8 = x^2 - 7\}$

3. What is the maximum number of pieces 1 foot 4 inches in length that you can cut from a board 9 feet 6 inches long?

4. If a man has x dollars and spends one third of this, how much does he have left?

5. If $12t + 8$ is the perimeter of a square, what is the side-measure?

6. If $x = 6$ and $y = 3$ then $\frac{4}{x} + \frac{1}{y} =$ ________.

7. A telephone bill is $4.40 including a 10% tax. What would the bill be if the tax were 5%?

8. If $x = -2$ and $y = -4$ then $\frac{1}{x} + \frac{6}{y} =$ ________.

9. If it takes 3d days for 3c cats to catch 3r rats, how many cats would it take to catch 21r rats in 21d days?

10. $x - y$ exceeds its opposite by how much?

L. 1. How many numbers are there whose square is 25?

2. How many positive numbers are there whose square is 144?

3. How many positive numbers are there whose square is 701?

4. $\forall_{p \geq 0} \sqrt{p} \geq$ ________

5. $\sqrt{25} \times \sqrt{25} =$ ________

6. $\sqrt{9} \times \sqrt{9} =$ ________

7. $\sqrt{144} \times \sqrt{144} =$ ________

8. $\sqrt{81} \times \sqrt{81} =$ ________

9. $(\sqrt{36})^2 =$ ________

10. $\sqrt{30} \times \sqrt{30} =$ ________

11. $\sqrt{2} \times \sqrt{2} =$ ________

12. $\sqrt{783} \times \sqrt{783} =$ ______

13. $\sqrt{5163} \times \sqrt{5163} =$ ______

14. $\sqrt{971 \times 835} \times \sqrt{971 \times 835} = 971 \times$ ______

15. $\sqrt{589 \times 762} \times \sqrt{589 \times 762} =$ ______ $\times 762$

16. $\sqrt{347 \times 698} \times \sqrt{698 \times 347} =$ ______ $\times$ ______

17. $\sqrt{1873 \times 6952} \times \sqrt{1873 \times 6952} =$ ______ $\times$ ______

18. $(\sqrt{1873} \times \sqrt{1873})(\sqrt{6952} \times \sqrt{6952}) =$ ______ $\times$ ______

19. $(\sqrt{1873} \times \sqrt{6952})(\sqrt{1873} \times \sqrt{6952}) =$ ______ $\times$ ______

20. $572 \times 189 = (\sqrt{572} \times \sqrt{572})(\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})$

21. $64 \times 36 = (\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})(\sqrt{64} \times \sqrt{36})$

22. $64 \times 36 = \sqrt{64 \times 36} \times \sqrt{\_\_\_\_ \times \_\_\_\_}$

23. $\sqrt{5 \times 7} \times \sqrt{5 \times 7} =$ ____ $\times$ ____

24. $(\sqrt{5} \times \sqrt{7}) \times (\sqrt{5} \times \sqrt{7}) =$ ____ $\times$ ____

25. How many positive numbers are there whose square is 5×7?

26. $\sqrt{5 \times 7} = \sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_}$

27. $\sqrt{18} \times \sqrt{2} = \sqrt{\_\_\_\_ \times \_\_\_\_}$

28. $\forall_{p \geq 0}\ \sqrt{p}\sqrt{p} =$ ____

29. $\forall_{a \geq 0}\forall_{b \geq 0}\ (\sqrt{a}\sqrt{a})(\sqrt{b}\sqrt{b}) =$ ____

30. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ (\sqrt{a}\sqrt{b})(\sqrt{a}\sqrt{b}) =$ ____

31. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ \sqrt{a}\ \sqrt{b}$ is <u>the</u> positive number whose square is ____.

32. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ \sqrt{ab}\ \sqrt{ab} =$ ____

33. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ \sqrt{\_\_\_\_}$ is <u>the</u> positive number whose square is ab.

34. In view of the results in Exercises 31 and 33, if ab is nonnegative, how many positive numbers are there whose square is ab?
Hence, $\forall_{a \geq 0}\ \forall_{b \geq 0}\ \sqrt{a}\ \sqrt{b} =$ ____.

35. $\sqrt{75} = \sqrt{\_\_\_\_ \times \_\_\_\_} = \sqrt{25} \times \sqrt{3} =$ ____ $\times \sqrt{\_\_\_\_}$

36. $\sqrt{\_\_} = \sqrt{100 \times 5} = \sqrt{\_\_} \times \sqrt{5} = \_\_ \times \sqrt{5}$

37. $\sqrt{98} = \sqrt{\_\_ \times \_\_} = \sqrt{\_\_} \times \sqrt{\_\_} = \_\_ \times \sqrt{\_\_}$

38. $\sqrt{\_\_} = \sqrt{\_\_ \times \_\_} = \sqrt{\_\_} \times \sqrt{\_\_} = 2\sqrt{7}$

39. $\sqrt{\_\_} = \sqrt{\_\_ \times \_\_} = \sqrt{\_\_} \times \sqrt{\_\_} = 3\sqrt{11}$

40. $\sqrt{160} = \sqrt{\_\_ \times \_\_} = \sqrt{\_\_} \times \sqrt{\_\_} = \_\_\sqrt{10}$

41. $\sqrt{108} = \sqrt{\_\_ \times \_\_} = \sqrt{\_\_} \times \sqrt{\_\_} = 6\sqrt{\_\_}$

42. $\sqrt{63} = \sqrt{\_\_} \times \sqrt{\_\_} = \_\_ \times \sqrt{\_\_}$

43. $\sqrt{\_\_} = 11\sqrt{3}$ 44. $\sqrt{\_\_} = 4\sqrt{2}$

M. Simplify.

1. $\frac{\sqrt{18}}{12}$ [Solution. $\frac{\sqrt{18}}{12} = \frac{\sqrt{9 \cdot 2}}{12} = \frac{\sqrt{9}\sqrt{2}}{12} = \frac{3\sqrt{2}}{12} = \frac{\sqrt{2}}{4}$]

2. $\frac{6 + \sqrt{12}}{4}$ [Solution. $\frac{6 + \sqrt{12}}{4} = \frac{6 + \sqrt{4}\,\sqrt{3}}{4} = \frac{6 + 2\sqrt{3}}{4} =$

$\frac{2(3 + \sqrt{3})}{4} = \frac{3 + \sqrt{3}}{2}$]

3. $\frac{\sqrt{27}}{6}$ 4. $\frac{\sqrt{24}}{8}$ 5. $\frac{\sqrt{75}}{20}$ 6. $\frac{\sqrt{200}}{55}$

7. $\frac{16 + \sqrt{12}}{2}$ 8. $\frac{14 - \sqrt{28}}{2}$ 9. $\frac{36 + \sqrt{63}}{15}$

10. $\frac{-2 - \sqrt{88}}{6}$ 11. $\frac{-5 + \sqrt{49}}{2}$ 12. $\frac{12 - \sqrt{8}}{4}$

13. $2\sqrt{2} + 3\sqrt{2}$ 14. $2\sqrt{2} + \sqrt{18}$ 15. $5\sqrt{3} + \sqrt{27}$

16. $3\sqrt{20} - 4\sqrt{5}$ 17. $\frac{2\sqrt{3}}{\sqrt{3}}$ 18. $\frac{\sqrt{12}}{\sqrt{3}}$

19. $\frac{\sqrt{45}}{3\sqrt{5}}$ 20. $\frac{2\sqrt{7} + \sqrt{63}}{\sqrt{7}}$ 21. $\frac{\sqrt{48} + \sqrt{27}}{\sqrt{3}}$

22. $\left(\frac{\sqrt{12}}{\sqrt{2}}\right)^2$ 23. $\frac{\sqrt{12}}{\sqrt{2}}$ 24. $\left(\frac{\sqrt{35}}{\sqrt{7}}\right)^2$ 25. $\frac{\sqrt{35}}{\sqrt{7}}$

26. $\left(\frac{\sqrt{121}}{\sqrt{11}}\right)^2$ 27. $\frac{\sqrt{121}}{\sqrt{11}}$ 28. $\frac{\sqrt{93}}{\sqrt{31}}$ 29. $\frac{\sqrt{51}}{\sqrt{17}}$

*

30. $\forall_{a \geq 0} \forall_{b > 0} \sqrt{\frac{a}{b}} = \sqrt{\underline{\quad} \cdot \frac{1}{b}} = \sqrt{\underline{\quad}} \cdot \sqrt{\frac{1}{b}} = \sqrt{a} \cdot \frac{\underline{\quad}}{\underline{\quad}} = \frac{\underline{\quad}}{\underline{\quad}}$

*

31. $\frac{6\sqrt{55}}{3\sqrt{11}}$ 32. $\frac{7\sqrt{15}}{14\sqrt{5}}$ 33. $\sqrt{\frac{144}{169}}$ 34. $3\sqrt{\frac{7}{81}}$

35. $\frac{3\sqrt{243}}{3\sqrt{27}}$ 36. $\frac{4\sqrt{51}}{12\sqrt{17}}$ 37. $24\sqrt{\frac{10}{72}}$ 38. $\frac{\sqrt{42}}{\sqrt{14}}$

39. $12\sqrt{\frac{14}{128}}$ 40. $\frac{3\sqrt{6}}{5\sqrt{2}}$ 41. $\frac{2\sqrt{21}}{6\sqrt{28}}$ 42. $\frac{\sqrt{6} + \sqrt{15}}{\sqrt{3}}$

43. $2\sqrt{128} + \sqrt{320}$ 44. $5\sqrt{2} \times 2\sqrt{18}$

45. $\frac{1817}{\sqrt{1817}}$ 46. $\frac{91\sqrt{70}}{13\sqrt{35}}$ 47. $\frac{21}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

N. Expand.

1. $\left(\frac{3 + \sqrt{5}}{2}\right)^2$ [Solution. $\frac{9 + 6\sqrt{5} + 5}{4} = \frac{14 + 6\sqrt{5}}{4} = \frac{7 + 3\sqrt{5}}{2}$]

2. $\left(\frac{5 + \sqrt{3}}{2}\right)^2$ 3. $\left(\frac{-2 + \sqrt{6}}{2}\right)^2$ 4. $\left(\frac{-3 - \sqrt{7}}{4}\right)^2$

5. $\left(\frac{3 + 2\sqrt{6}}{2}\right)^2$ 6. $\left(\frac{3 - 2\sqrt{6}}{2}\right)^2$ 7. $\left(\frac{-1 - 2\sqrt{3}}{2}\right)^2$

*

8. Is $1 + \sqrt{6}$ a root of '$x^2 - 2x - 5 = 0$'?

9. Show that $\frac{5 + \sqrt{89}}{4}$ and $\frac{5 - \sqrt{89}}{4}$ satisfy '$2x^2 - 5x - 8 = 0$'.

O. 1. 20 increased by 10% is _______.

[Solution. $20 + 10\%(20) = 20 + 20 \cdot 0.1 = 20 + 2 = 22$]

2. 40 increased by 25% is _______.

3. 75 increased by 100% is _______.

4. 200% of 75 is _______.

5. 16 is _______% of 20.

6. 45 is _______% of 30.

7. H increased by 50% is _______.

8. 100 increased by x% is _______.

9. m increased by n% is _______.

10. a is _______% of 8.

11. 36 is _______% of d.

12. _______ increased by 50% is 6.

13. 40 increased by _______% is 52.

14. f is _______% of g.

15. _______ is j% of k.

16. p is r% of _______.

17. Jim missed 20% of the problems on his math test. He got 96 problems right. How many problems were there on the test?

18. Allan entered the math contest again this year and scored 28% better than he did last year. His score this year was 32. What was Allan's score last year?

19. If the list price of an article is \$50, and the discount [or: discount rate] is 20% then the purchase price of the article is 50 - 20%(50), or 40, dollars. Complete the following table.

List price	50	72	100		40	78	60	320
Discount rate	40%	$12\frac{1}{2}\%$		30%		$37\frac{1}{2}\%$		
Purchase price			75	119	15		58	300

20. Wholesale merchants sometimes offer special inducements to get purchasers to buy large quantities of an item. One such inducement involves chain discounts. For example, if one purchases at least one thousand of the items, he gets a discount of 20% but if he purchases more than five thousand, he gets discounts of 20% and 2%. So, if the list price is \$1, and the purchaser buys 3000 articles, he pays 3000 - 20%(3000), or 2400, dollars. But, if he buys 7000, the purchase price is computed as follows.

$$7000 - 20\%(7000) = 5600$$
$$5600 - 2\%(5600) = 5488$$

So, he pays \$5488. [Is this what he would pay if the discount rate were a straight 22%?]

Complete the following table.

	List price	Chain discount	Purchase price
(a)	100	30%, 10%	____
(b)	500	20%, 5%	____
(c)	500	5%, 20%	____
(d)	1000	16%, 4.5%	____
(e)	3200	$12\frac{1}{2}$%, $2\frac{1}{2}$%	____
(f)	____	10%, 5%	513
(g)	____	15%, 1%	572.22
(h)	800	40%, __%	336

21. Jack bought a current model foreign car for \$1600. The list price was discounted 20% because the new model had just come out. The price of the new model was 20% more than the list price of the current model. What was the list price of Jack's car?

22. A storekeeper marked down a \$40 portable radio 20%. If he offered you an additional 25% discount from the new price, how much would he want for the radio?

23. A man bought 2 shares of stock and sold them a month later for \$60 each. On one share he made a profit of 20%, and on the other he took a loss of 20%. Did he gain, break even, or lose on the transaction? If he didn't break even, how much was his gain or loss?

P. 1. 2 dollars = ____ cents

2. 4 yards = ____ feet

3. 3 quarts = ____ pints

4. 16 decimeters = ____ centimeters

5. 24 pints = ____ gallons

6. 3.3 minutes = ____ seconds

7. 3 inches = ____ centimeters

8. 254 centimeters = ____ inches

9. 1 foot = ____ centimeters

10. 1 meter = ____ inches

11. 1 yard = ____ meters

12. 1 kilometer = ____ centimeters

13. 1 mile = ____ kilometers

14. 3 decimeters = ____ yards

15. 1 hour = ____ seconds

16. 1.5 kilometers = ____ meters

17. 2 centimeters = ____ decimeters

18. 35 millimeters = ____ decimeters

19. 3 feet = ____ centimeters

20. 2.5 quarts = ____ pints

21. 1 pint = ____ quarts

22. 1 foot = ____ meters

23. 0.1 millimeters = ____ centimeters

24. 1 square foot = ____ square inches

25. 72 square inches = ____ square feet

26. 1 square inch = ____ square centimeters

27. 55 square yards = ____ square feet

28. 43 square centimeters = ____ square inches

29. 278 square centimeters = ____ square meters

30. 2 square meters = ____ square decimeters

31. 100 cubic yards = ____ cubic feet

32. 3456 cubic inches = ____ cubic feet

33. 1000 cubic inches = ____ cubic centimeters

34. 1 cubic mile = ____ cubic yards

35. 62 cubic centimeters = ____ cubic meters

36. d days = ____ weeks

37. p pounds = ____ ounces

38. p pounds = ____ tons

39. s seconds = ____ hours

40. m miles = ____ inches

Q. The number 47 has many names. A few of them are '$9 \times 5 + 2$', '$40 + 7$', and '$2 \times 3 \times 7 + 5$'. The simplest looking of all is '47'. This numeral is called the decimal numeral for 47. It is an abbreviation for '$4 \cdot 10^1 + 7$'. Similarly, the decimal numeral for 603 is '603', and this is an abbreviation for '$6 \cdot 10^2 + 0 \cdot 10^1 + 3$'. The fact that we use powers of ten in developing the name for the number explains the use of the word 'decimal'.

1. If the digits in '43' are reversed, what number is named?

2. If you reverse the digits in the decimal numeral for 79567, what number is named?

3. Repeat Exercise 2 for 68086. For 99.

4. What is the sum of the digits of the decimal numeral for 342? [That is, what is the sum of the numbers named by '3', '4', and '2'?]

5. What is the sum of the digits in that whole number between 71 and 79 which is exactly divisible by 5? [This is a quick way of asking: What is the sum of the numbers named by the digits in the decimal numeral for that whole number between ... ?]

6. The sum of the digits in a two-digit number [a two-digit number is one whose decimal numeral consists of two digits] is 9. If the number is between 50 and 60, what is it?

7. The sum of the digits in a two-digit number is 7. If you reverse the digits and add this number to the original number, what sum do you get?

8. Pick a two-digit number which has different digits. [For example, 47 but not 33.] Reverse the digits to get a second number, and subtract the smaller of the two numbers from the larger. Reverse the digits in the difference, and add this number to the difference.

9. Repeat Exercise 8 for a new two-digit number, and compare the sum you get now with the one you got in Exercise 8.

10. Pick any number such that the sum of its digits is [exactly] divisible by 3. Is the number itself divisible by 3?

R. 1. This chart shows the distribution of scores on a test. For example, two people scored 20 points, three scored 19 points, etc.

Score	Frequency
20	II
19	III
18	~~IIII~~ I
17	~~IIII~~ IIII
16	III
15	~~IIII~~
14	
13	II
12	I

(a) How many scored 16?

(b) How many scored 14?

(c) How many people took the test?

(d) How many scored at least 16?

(e) How many scored above 17?

(f) How many scored below 17?

(g) How many scored no better than 17?

(h) How many scored no worse than 17?

(i) If the students scores were arranged in order from the highest to the lowest, which score would be first?

(j) Which would be thirty-first?

(k) Which would be fourth? (ℓ) Which would be fifth?

(m) Which score is the middle score? [This is called the median score.]

(n) Which score was obtained by the largest number of students? [This is called the modal score.]

(o) What is the average for the distribution? [The word 'average' is commonly used to refer to the arithmetic mean [pronounced as 'a-rith-met'-ic mean'] of the scores. You compute it by multiplying each score by its frequency, adding the products, and dividing by the total number of people.]

2. Here is another distribution of test scores.

Score	3	4	5	6	7	8	9	10
Frequency	1	2	0	5	0	4	3	1

(a) What is the median score? Did anyone get this score?

(b) What is the modal score?

(c) What is the arithmetic mean score?

3. Pete's bowling scores so far this year are 132, 97, 122, 145, 117, 136, 128, 102, 169, 121, and 137.

   (a) What is his median score?

   (b) What is the arithmetic mean score?

S. 1. The sum of two numbers is 7, and one of them is 3 more than the other. What are they?

2. The difference of one number from another is 5, and their sum is 76. What are the numbers?

3. A number exceeds another by 11 and their sum is 19. What are the numbers?

4. The difference of one number from another is 6, and their product is 16. What are the numbers?

☆ 5. If a first number is added to 51, the sum is 10 times a second number. If the second number is subtracted from 51, the difference is 5 times the first number. What are the numbers?

6. Al is now 3 times as old as his young cousin Bill, and in 8 years he will be twice as old. How old are Al and Bill now?

7. 13 years ago Kathy was twice as old as Cheryl. The square of the sum of their ages now is 961. What are their ages now?

8. An envelope contains 2 coins worth 60¢. However, one of the coins is not a dime. What are the coins?

9. Leslie found a chest containing just quarters, half-dollars, and silver dollars. There were twice as many silver dollars as half-dollars, and twice as many half-dollars as quarters. The chest contained 91 coins. What is the value of all Leslie's coins?

10. Carson has $7.92 worth of pennies, nickels, and dimes in his pocket. He has 3 times as many pennies as dimes and one-fourth as many dimes as nickels. How many of each coin does Carson have in his pocket?

11. A farmer has 2 cows in his north pasture. If he puts 37 sheep in the same pasture, how many cows will then be in the pasture?

12. A drug manufacturer has a 50-gallon barrel holding 20 gallons of a 10% alcohol solution. If he adds 17.5 gallons of water, how many gallons of alcohol will he then have in the barrel?

13. David has 55 milliliters of a 40% sulphuric acid solution. If he adds 5 ml. of pure acid to his solution, how many ml. of pure water will it contain? How many ml. of solution will there be? What percent of the new solution is water? What percent (of the new solution) is acid?

14. How much ginger ale must be added to 1 gallon of fruit juice to make a punch that is 40% ginger ale?

15. A one gallon solution consists of 5% Lozak and 95% Bazol. How much Lozak must be added to get a new solution which is 50% Bazol?

16. How much Bazol must be added to a quart of a 15% Bazol solution to get one that is a 20% solution?

17. How much water must be added to 50 gallons of a 10% Lozak solution to get one that is an 8% solution?

18. How many gallons of a 20% Lozak solution must be added to 2 quarts of a 6% Lozak solution to get a solution that is 10% Lozak?

19. How many gallons of a 39% Bazol solution must be added to 16 pints of a 51% Bazol solution to give a 60% Bazol solution?

20. If 1 pound of candy sells for $1.37, what is the price of 52 pounds of the same candy?

21. If x pounds of tea costs y cents, what is the price of 1 pound of tea? What is the cost of w pounds of tea?

22. John sold 5 pounds of 65¢-a-pound grass seed for 40¢ a pound. How many pounds of this seed must John sell at 80¢ a pound to make up for his error?

23. A candy factory makes fruit and nut chocolates to sell for $1.28 per pound. Their creams sell for $1.80 per pound. If 100 pounds of a mixture is to be worth $1.56 per pound, how many pounds of creams should it contain?

24. Don can paddle his blue canoe down an 8-mile stream in 2 hours, but it takes him 6 hours to paddle back upstream.

    (a) What is his average rate of speed for a round trip?

    (b) What is the rate of flow of the stream?

25. The new pump can fill Ginny's swimming pool in 4 hours, but it takes the old one 12 hours to do the same job. How long would it take both pumps to fill the pool if they operated together?

T. Solve each of the given equations for the indicated variable.

1. $3x + y = 7$; y
2. $y - 5x = 9$; y
3. $2y - 8x = 12$; y
4. $3y + 6x = 10$; y
5. $2a + 5b = 7$; b
6. $tx = s + nx$; x
7. $3m - n - 8 = 0$; n
8. $3m - n - 8 = 0$; m
9. $ax + y + c = 0$; y
10. $ax + by + c = 0$; y
11. $my + ny = k$; y
12. $10r + 7s - 5 = 0$; s
13. $5y = 10z + 5$; y
14. $2y + 8 = 12x - y + 3$; y
15. $5(x - 3) + 7(y - 4) = 0$; y
16. $9(5 - a) - (b - 7) = 0$; b
17. $kp - mp = k^2 - m^2$; p
18. $tx = t^2 - s^2 - sx$; x
19. $y(x - 3) + x(5 - y) = 0$; x
20. $3(2x - 4y + 1) - 2(3x + y - 5) = 0$; y

U. Simplify.

1. $3a + 7 - a$
2. $10x - 8x + 7$
3. $5k + 2m - 3k - 3m$
4. $6y + 7x - 6x - y$
5. $3x^2 + 2x + 5x^2 - 8x$
6. $2a^2 - 3b^2 + 9a^2 + b^2 - a^2$
7. $5ab - 7bc + 2ab - bc$
8. $12xy - 3x + 4xy - 7y$
9. $-mn + 6m^2 - 3n + 2mn$
10. $6p^2 - 3pq + q^2 - 2p^2 + 2q^2$
11. $6(a - 2b - 5) - 3(3a + 4b)$
12. $5(x - 3y - 1) + 7(2x + y - 3)$
13. $7(m + 2n) - (5m - n)$
14. $-(x - 2y - z) - 4(-x + y - 3z)$
15. $x(2x - 7) - 3x(4 - 2x)$
16. $y(1 - 8y) - 2y(1 + 3y) - (3 - y)$

17. $x^2(1 - x) + x(3 - 2x + 3x^2)$

18. $3x(2y - x) - 5y(8x - 3y + 1)$

19. $6a(a - b) + 7b(-a + 2b)$

20. $2x^2(x^2 - 3x + 1) - 5x^2(-x^2 + 2x - 3)$

21. $5a(3b)(-2c)$

22. $x(-2y)(-3z)$

23. $5m(-2a)(7b)$

24. $8x(2x)^2(-3y)^2$

25. $0.1a(1.1a)(8a)$

26. $-k(3k)(-2k^2)$

27. $4a^2(3a^3)(-2a^4)$

28. $9b^3(-b^2)(-2b)$

29. $x^2y(3xy^2)(2xy)$

30. $2ab(3a^2b^3)(-ab^3)$

31. $5m(3m^2n)^2(2mn^2)^3$

32. $-3a(-2a^2b)^3(-a^3b^3)^2$

33. $5xy^2(2x^2y) - 7x^3y^3$

34. $2a^2b^2(-3ab^2) - 4ab^3(-2a^2b)$

35. $6a^2b^2(ab + 5) - 3ab^3(2a^2)$

36. $7x^2y^4(-2xy^3 + 1) - 3xy(x^2y^6 - 2xy^3)$

37. $\dfrac{5ab}{10ab}$

38. $\dfrac{3x^2y}{3xy^2}$

39. $\dfrac{x^2y^3z^2}{xy^4z^2}$

40. $\dfrac{(ab^2)(ac)^3}{a^4b^2c^2}$

41. $\dfrac{-5xy}{-x(x^2y)}$

42. $\dfrac{3x(-2y)^2}{(-3x)^2(4y^3)}$

43. $\dfrac{3x^2(a + b)^2}{9x(a + b)}$

44. $\dfrac{(c - d)^3}{(d - c)^2}$

45. $\dfrac{1}{3} + \dfrac{2}{x} + \dfrac{2}{5}$

46. $\dfrac{x}{4} - \dfrac{3}{x} + \dfrac{1}{2}$

47. $\dfrac{y}{2} - \dfrac{5}{x} + \dfrac{1}{3x}$

48. $\dfrac{2}{xy} - \dfrac{3}{y} + \dfrac{4}{x}$

49. $\dfrac{3}{x^2} - \dfrac{7}{x} + \dfrac{5}{2x}$

50. $\dfrac{1}{x^2y} - \dfrac{3}{xy^2} + \dfrac{2}{xy}$

51. $\dfrac{2}{x+3} + \dfrac{5}{x+2}$

52. $\dfrac{4}{y-2} + \dfrac{3}{y+5}$

53. $\dfrac{8}{z-1} - \dfrac{5}{z-2}$

54. $\dfrac{1}{(x+3)^2} + \dfrac{5}{x+3}$

55. $\dfrac{3}{x+4} + \dfrac{1}{(x+4)(x-1)}$

56. $\dfrac{7}{(x-5)(x+1)} + \dfrac{3}{x+1}$

57. $\dfrac{4}{x^2 + 5x + 6} + \dfrac{3}{x + 2} + \dfrac{5}{x + 3}$

58. $\dfrac{7}{y-4} - \dfrac{3}{y + 4} + \dfrac{2}{y^2 - 16}$

59. $\dfrac{2a + 3b}{a - b} + \dfrac{a + b}{a + 3b} - \dfrac{6a - 5b}{a^2 + 2ab - 3b^2}$

60. $\dfrac{2}{z - 3} + \dfrac{5}{z^2 + 2z - 15} - \dfrac{1}{z + 5}$

61. $\dfrac{3xy^2}{2ab} \cdot \dfrac{5a^2b^3}{7x^2y}$

62. $\dfrac{4a^2b^5}{9xy^3} \cdot \dfrac{3(xy)^2}{8(a^2b)^3}$

63. $\dfrac{2(x-5)^2}{3(y-7)^2} \cdot \dfrac{-(y-7)^3}{6(x-5)^4}$

64. $\dfrac{(x-2)(x+3)}{(y-4)(y+7)} \cdot \dfrac{(y-4)(y+5)}{(x+3)(x-4)}$

65. $\dfrac{x^2 + x - 6}{y^2 + 3y - 28} \cdot \dfrac{y^2 + y - 20}{x^2 - x - 12}$

66. $\dfrac{a^2 - 2a - 8}{b^2 - 8b + 15} \cdot \dfrac{b^2 + b - 30}{a^2 - 16}$

67. $\dfrac{a^3 - 4a^2 - 21a}{a^4 - 2a^3 - 35a} \cdot \dfrac{a^5 + 8a^4 + 15a^3}{a^3 + 6a^2 + 9a}$

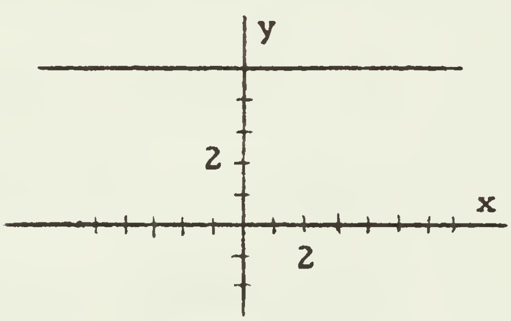
SUPPLEMENTARY EXERCISES

A. Draw graphs of these relations. [Label the axes in accordance with the conventions adopted in Unit 4.]

1. $\{(x, y): y = 3x - 2\}$
2. $\{(x, y): y - 3x = 4\}$
3. $\{(a, b): a^2 + b^2 = 25\}$
4. $\{(s, t): |s| + |t| \leq 10\}$
5. $\{(u, v): v = 8\}$
6. $\{(u, v): u = 8\}$
7. $\{(a, b): a \geq 2 \text{ or } b \leq 3\}$
8. $\{(a, b): a \not\geq 2 \text{ and } b \not\leq 3\}$
9. $\{(x, y), x \neq 0: y = x - \frac{x}{x}\}$
10. $\{(x, y): y = x^3\}$
11. $\{(x, y): x^2 \neq 25\}$
12. $\{(x, y): 2x + y \leq 6\}$
13. $\{(x, y): (x \in I \text{ and } y = 3) \text{ or } (x \notin I \text{ and } y = -1)\}$
14. $\{(x, y): |3 - x| \leq 2 \text{ and } |y - 4| \leq 2\}$
15. $\{(x, y): |x| + |y| \geq |x + y|\}$
16. $\{(x, y): xy = 12\}$
17. $\{(x, y): x > 5 \text{ and } |y| + x = 3\}$
18. ☆ $\{(x, y): (x - 2)(x - 3) \geq 0\}$
19. ☆ $\{(x, y): xy - x^2 = 0\}$
20. ☆ $\{(x, y): x^2 - 7x + 10 \leq y\}$

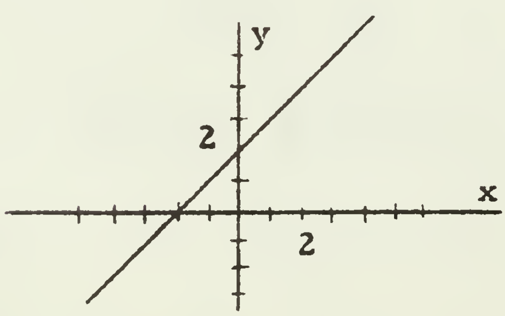
B. In each exercise you are given a graph of a relation and five names of relations. Encircle the letter preceding a name of the pictured relation. <u>There may be more than one correct choice.</u>

1.



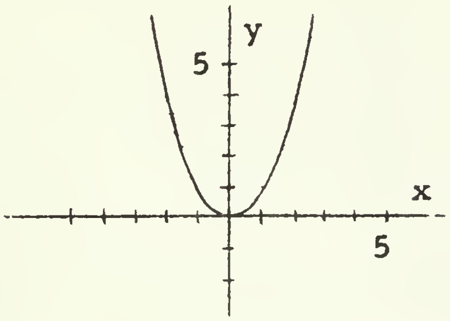
(a) $\{(x, y): x = 5\}$
(b) $\{(x, y): y = 5x\}$
(c) $\{(x, y): y = 5\}$
(d) $\{(x, y): y = 5 \text{ and } x = 2\}$
(e) $\{(x, y): 2y - 2 = 3 + y\}$

2.



(a) $\{(x, y): y = x - 2\}$
(b) $\{(x, y): y = x + 2\}$
(c) $\{(x, y): 2y = x + 2\}$
(d) $\{(x, y): 3y = 3x + 6\}$
(e) $\{(x, y): y = 5x + 2\}$

3.



(a) $\{(x, y): y = x^2\}$

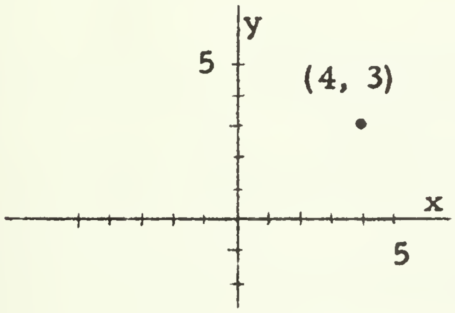
(b) $\{(x, y), y \geq 0: \sqrt{y} = x\}$

(c) $\{(x, y): y = |x|\}$

(d) $\{(x, y): y^2 = x\}$

(e) $\{(x, y): y = 2x\}$

4.



(a) $\{(x, y): x = 4 \text{ or } y = 3\}$

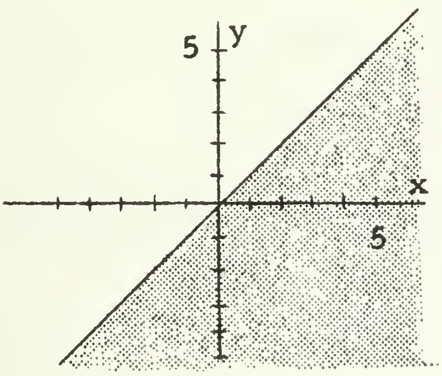
(b) $\{(x, y): y = 3 \text{ and } x = 4\}$

(c) $\{(x, y): y - x = 1 \text{ and } y + 2x = 10\}$

(d) $\{(x, y): y - 2x = -5 \text{ and } 3x - y = 9\}$

(e) $\{(x, y): y = x + 1 \text{ and } x = 3\}$

5.



(a) $\{(x, y): y > x\}$

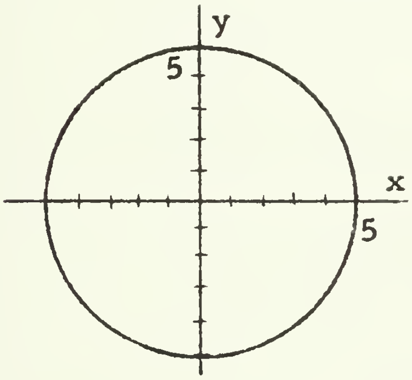
(b) $\{(x, y): y \leq x\}$

(c) $\{(x, y): x \geq y\}$

(d) $\{(x, y): x \geq 0\}$

(e) $\{(x, y): y - x \text{ is nonpositive}\}$

6.



(a) $\{(x, y): x^2 - y^2 = 25\}$

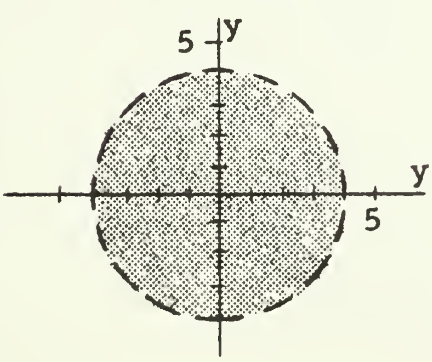
(b) $\{(x, y): x^2 = 25 - y^2\}$

(c) $\{(x, y): \sqrt{2}\, x^2 + \sqrt{2}\, y^2 = \sqrt{1250}\}$

(d) $\{(x, y): |x| + |y| = 5\}$

(e) $\{(x, y): x^2 + y^2 = 25\}$

7.



(a) $\{(x, y): x^2 + y^2 \leq 16\}$

(b) $\{(x, y): x^2 + y^2 > 16\}$

(c) $\{(x, y): x^2 + y^2 < 16\}$

(d) $\{(x, y): x < 4 \text{ and } y < 4\}$

(e) $\{(x, y): y < 4 \text{ and } x^2 + y^2 = 4\}$

C. 1. Graph the relations A, B, C, D, and E where

$A = \{(x, y): y - x = 5\}$, $B = \{(x, y): y + 2x = 2\}$,

$C = \{(x, y): y - 3x \leq 7\}$, $D = \{(x, y): 1 < x < 5\}$,

$E = \{(x, y): |y| \leq 2\}$.

2. Graph and find the ordered pairs in $A \cap B$.

3. Graph $B \cup C$.

4. Graph $C \cup E$.

5. Graph and list the elements in $\{(x, y) \in I \times I: (x, y) \in D \cap E\}$.

6. Graph $A \cap C$; $B \cap C$; $(A \cap C) \cup (B \cap C)$; $(A \cup B) \cap C$.

7. Graph $B \cap E$; $B \cap D$; $B \cap (D \cap E)$; $B \cap D) \cap E$.

8. Graph $C \cap E$; $C \cap D$.

9. Graph $\tilde{C}$; $\tilde{D}$; $\tilde{E}$; $\tilde{A}$.

10. Graph $\tilde{D} \cap \tilde{E}$; $\tilde{D} \cup \tilde{E}$; $\widetilde{D \cup E}$; $\widetilde{D \cap E}$.

11. Graph $(D \cap E) \cup \tilde{C}$.

12. Graph $[C \cap (\widetilde{D \cap E})] \cap B$.

13. Graph $M \cap N$ where

$M = \{(x, y): x^2 + y^2 \leq 25\}$

$N = \{(x, y): x^2 + y^2 > 9\}$.

14. Graph $\{(x, y): x^2 + y^2 \leq 16\} \cap \{(x, y): |x| > 2\}$.

☆ 15. Graph $\{(x, y): y^2 - x^2 = 0\} \cap \{(x, y): xy > 0\}$.

☆ 16. Graph $A \cup B \cup C \cup D \cup E \cup F \cup G$ where

$A = \{(x, y): x^2 + y^2 = 4\}$,

$B = \{(x, y): x = 0 \text{ and } -9 \leq y \leq -2\}$,

$C = \{(0.5, 1), (-0.5, 1)\}$

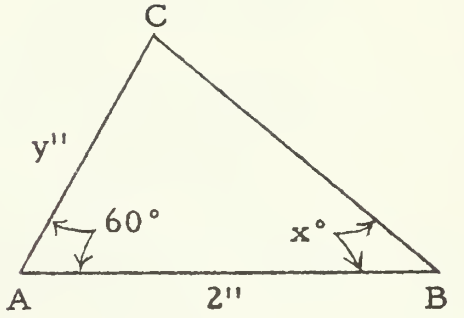
$D = \{(x, y): |x| = 1/16 \text{ and } y = -1/8\}$

$E = \{(x, y): y < -1/2 \text{ and } x^2 + y^2 = 1\}$

$F = \{(x, y): |x| \leq 2 \text{ and } y = (-3/2)|x| - 9\}$

$G = \{(x, y): y = -4 \text{ and } |x| \leq 4\}$

D. 1. Consider a triangle, ΔABC, in which $\angle A$ is an angle of 60° and the inch-measure of $\overline{AB}$ is 2. Draw a graph of the relation of the inch-measure of side $\overline{AC}$ to the degree-measure of $\angle B$.



2. Use the graph to help you complete the following:

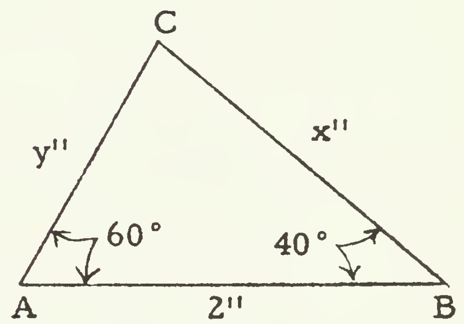
(a) If $\angle B$ is an angle of 30° then the inch-measure of $\overline{AC}$ is ___.

(b) If $\angle B$ is an angle of 60° then the inch-measure of $\overline{AC}$ is ___.

(c) If $\angle C$ is an angle of 30° then the inch-measure of $\overline{AC}$ is ___.

(d) The degree-measure of $\angle B$ must be a number between ______ and ______.

3. (a) Consider a triangle, ΔABC, in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and $\angle B$ is an angle of 40°. Draw a graph of the relation of the inch-measure of $\overline{AC}$ to the inch-measure of $\overline{BC}$. Interpret your results.



(b) Now consider a triangle, ΔABC, in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and the inch-measure of $\overline{AC}$ is 3. What can you say about the relation of the inch-measure of $\overline{BC}$ to the degree-measure of $\angle B$? Interpret your results in terms of the variety of triangles you can draw which fit the three conditions given for ΔABC.

4. Consider a triangle, ΔABC, which fits the two conditions given in Exercise 1. Draw a graph of the relation of the inch-measure of $\overline{AC}$ to the inch-measure of $\overline{BC}$. [Be sure you include the graphs of points which correspond with triangles in which the

degree-measure of $\angle B$ is 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and 110.]

(a) How many differently-shaped triangles ABC can you draw in which $\angle A$ is an angle of 60°, the inch-measure of $\overline{AB}$ is 2, and the inch-measure of $\overline{BC}$ is 4?

(b) In which the inch-measure of $\overline{BC}$ is 1.8?

(c) In which the inch-measure of $\overline{BC}$ is 1?

E. 1. Give the domain and range of each relation in Part A of the Supplementary Exercises.

2. Give the domain, range, and field of each relation described below.

(a) $A = \{(x, y) \in I^+ \times I^+: x > y\}$

(b) $B = \{(x, y) \in I \times I: xy \text{ is odd}\}$

(c) $C = \{(x, y) \in I \times I: xy \text{ is even}\}$

(d) $D = \{(x, y) \in I \times I: x \text{ is odd and } x + y \text{ is odd}\}$

(e) $E = \{(x, y) \in I^+ \times I^+: y = x^2\}$

(f) $F = \{(x, y) \in I \times I: y = 13\}$

(g) $G = \{(x, y) \in I^+ \times I^+: y < 25 \text{ and } y = 36\}$

(h) $H = \{(x, y) \in I^+ \times I^+: x < 30 \text{ and } y \text{ is the sum of the } \underline{\text{proper}} \text{ factors of } x \text{ with respect to } I^+\}$

Note: With respect to I^+, a proper factor of a positive integer is one of its factors which is smaller than it.

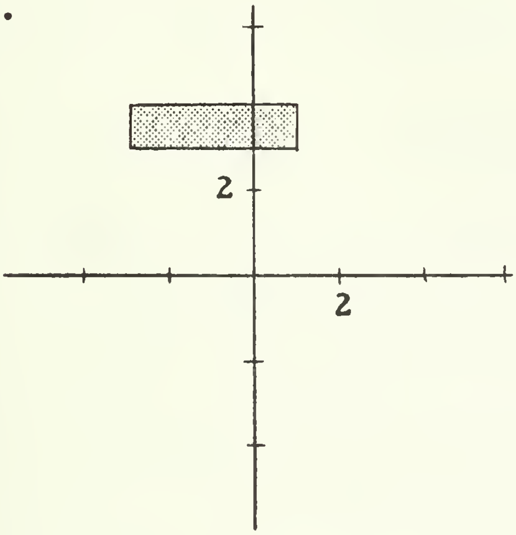
3. (a) If you know the domain and the range of a relation, can you use this information alone to tell what the field of the relation is?

(b) If you know the domain and the field of a relation, can you use this information alone to tell what the range of the relation is?

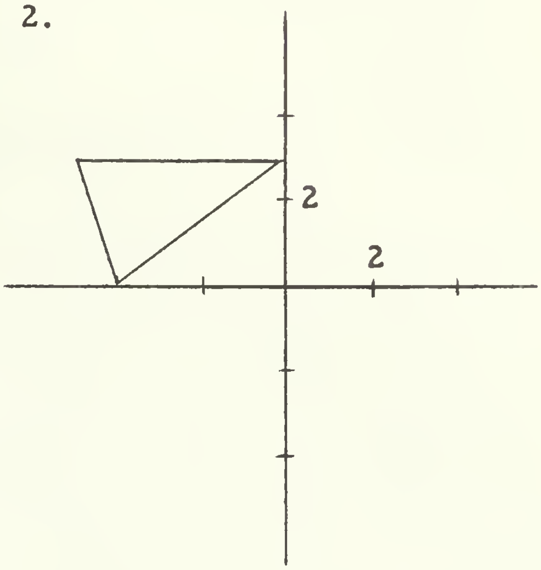
(c) If you know the domain and the range of a relation, can you use this information alone to tell what ordered pairs belong to the relation?

F. Each of the following exercises contains a graph of a relation. Sketch the graph of the converse of the relation.

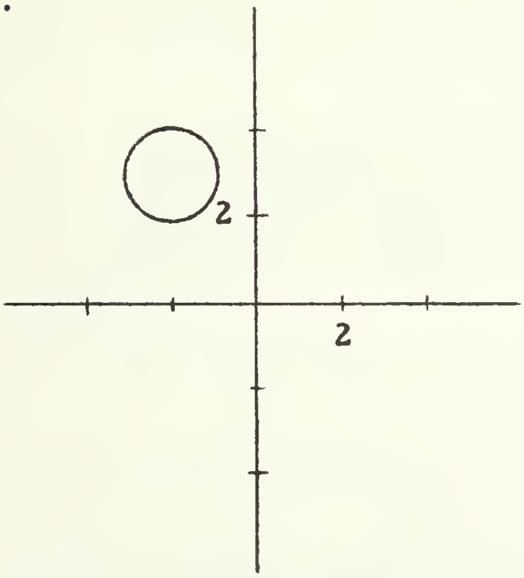
1.



2.



3.



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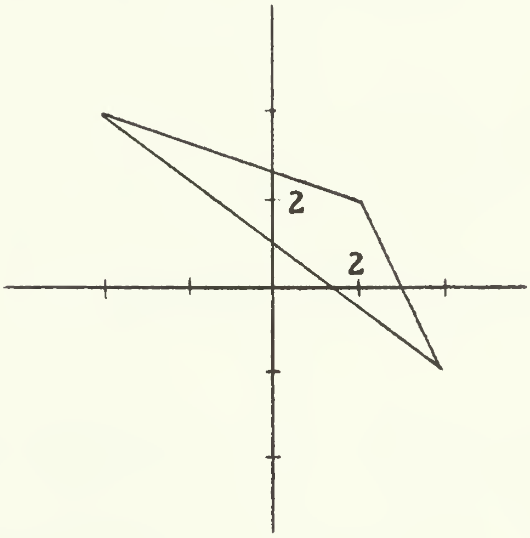

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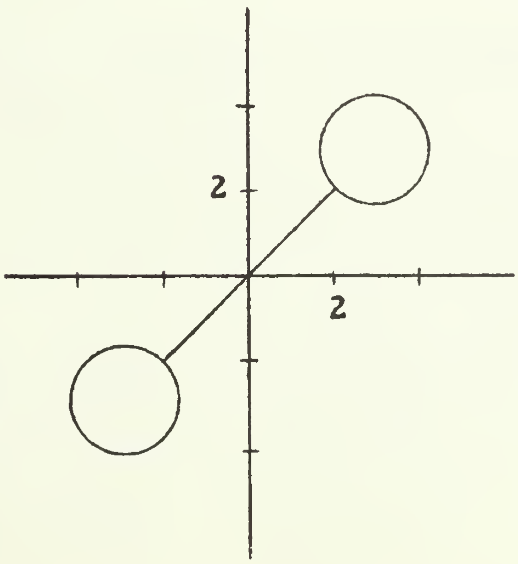

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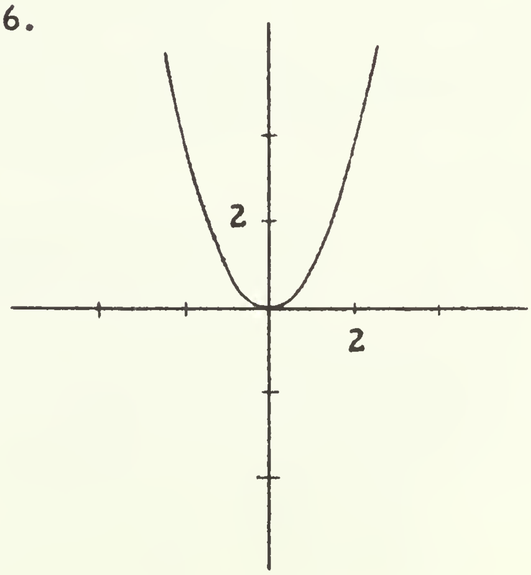


G. 1. Which of the relations given in Part A of the Supplementary Exercises are reflexive? Which are symmetric?

2. Which of the relations listed below are reflexive, and which are symmetric? [C is the set of all students in a Zabranchburg High mathematics class.]

(a) $\{(x, y) \in C \times C$: x is older than y$\}$

(b) $\{(x, y) \in C \times C$: x is in the same chemistry class as y$\}$

(c) $\{(x, y) \in C \times C$: x is a brother or sister of y$\}$

(d) $\{(x, y) \in C \times C$: x lives on the same street as y$\}$

(e) $\{(x, y) \in C \times C$: x has the same birthday as y$\}$

(f) $\{(x, y) \in C \times C$: x knows the address of y$\}$

3. There are many relations whose field is $\{1, 3, 5, 7, 9, 11, 13\}$ and which are reflexive. List the members of that one of these relations which has the fewest members.

4. Suppose R_1 and R_2 are two reflexive relations with the same nonempty field. Which of the relations listed below must be reflexive? [Each complement is relative to the cartesian square of the field.]

(a) $R_1 \cap R_2$ (b) $R_1 \cup R_2$ (c) $\tilde{R}_1$ (d) $\tilde{R}_1 \cap R_2$

(e) $\tilde{R}_1 \cup R_2$ (f) $\tilde{R}_1 \cap \tilde{R}_2$ (g) $\tilde{R}_2$ (h) $\tilde{R}_1 \cup \tilde{R}_2$

5. (a) Describe the intersection of all reflexive relations with $\{1, 2, 3, 4, 5\}$ as field.

(b) Describe the union of all reflexive relations with $\{1, 2, 3, 4, 5\}$ as field.

6. Suppose S_1 and S_2 are two symmetric relations among the members of a set M. Which of the relations listed below are symmetric? [Each complement is relative to $M \times M$.]

(a) $\tilde{S}_1$ (b) $\tilde{S}_2$ (c) $S_1 \cup S_2$ (d) $S_1 \cap S_2$

(e) $\tilde{S}_1 \cup \tilde{S}_2$ (f) $\tilde{S}_1 \cap \tilde{S}_2$ (g) $\widetilde{S_1 \cup S_2}$ (h) $\widetilde{S_1 \cap S_2}$

(i) $\widetilde{\tilde{S}_1 \cup \tilde{S}_2}$ (j) $\widetilde{\tilde{S}_1 \cap \tilde{S}_2}$ (k) $S_1 \cup \tilde{S}_2$ (ℓ) $\tilde{S}_1 \cap S_2$

7. Which of these relations are symmetric?

   (a) motherhood
   (b) sisterhood
   (c) $\{(x, y): x \neq y\}$
   (d) $\{(x, y): x \not> y\}$
   (e) $\{(x, y): |x| + |y| < |x + y|\}$
   (f) $\{(x, y) \in E \times E$: x is equivalent to y$\}$ where E is the set of all equations
   (g) $\{(x, y) \in L \times L$: x is parallel to y$\}$ where L is the set of all straight lines
   (h) $\{(x, y) \in L \times L$: x is perpendicular to y$\}$
   (i) $\{(x, y) \in S \times S: x \subseteq y\}$ where S is the set of all subsets of $\{1, 2, 3, 4\}$

8. Recall that we say that the set of integers is <u>closed</u> under the operation addition because each sum of integers is an integer.

   (a) Consider the set of all symmetric relations among members of a set S. Under which of the operations unioning, intersecting, and complementing is this set of relations closed?

   (b) Repeat part (a) for the set of all reflexive relations among the members of a set S.

☆ <u>H.</u> Consider the relation $>$ among the real numbers. The field of this relation is the set of real numbers. This relation is not reflexive [that is, it is <u>nonreflexive</u>] because there is a member $k \in \mathfrak{F}_>$ such that $k \not> k$. [7 is such a member.] In fact, for each member $k \in \mathfrak{F}_>$, $k \not> k$. A relation which has this latter property is said to be an <u>irreflexive</u> relation.

A relation R is irreflexive if and only if

$$\forall_{x \in \mathfrak{F}_R} (x, x) \notin R.$$

1. Classify the relations listed below into three categories--reflexive, irreflexive, and nonreflexive but not irreflexive.

   (a) $\{(x, y) \in P \times P$: x is the grandfather of y$\}$
   (b) $\{(x, y): x + y = y + x\}$
   (c) $\{(x, y): xy = 0\}$

(d) $\{(x, y): x^2 + y^2 = 17\}$ (e) $\{(x, y): x \leq y\}$

(f) $\{(x, y): y = x^2 - 42\}$ (g) $\{(x, y): y = |x|\}$

(h) $\{(x, y) \in L \times L$: x is perpendicular to y$\}$, where L is the set of all straight lines

(i) $\{(x, y) \in I \times I$: x and y have a common integral factor$\}$

(j) $\{(x, y): x^2 + y^2 = -17\}$

2. Suppose R is a reflexive relation and N is an irreflexive relation and $\mathfrak{F}_R = \mathfrak{F}_N \neq \emptyset$. Classify the relations listed below into two categories--reflexive and irreflexive. [Complements are with respect to the cartesian square of the common field.]

(a) $N \cap R$ (b) $N \cup R$ (c) $\tilde{N}$ (d) $\tilde{R}$ (e) $\tilde{N} \cap R$

(f) $\tilde{N} \cup R$ (g) $N \cup \tilde{R}$ (h) $\widetilde{N \cup \tilde{R}}$ (i) $N \cap \tilde{R}$ (j) $\widetilde{N \cap \tilde{R}}$

3. Consider the relations F, B, and M where

$F = \{(x, y) \in P \times P$: y is the father of $x\}$,

$B = \{(x, y) \in P \times P$: y is a brother of $x\}$,

and $M = \{(x, y) \in P \times P$: y is the spouse of $x\}$.

(a) Which of the following statements are true?

(i) For each $(x, y) \in P \times P$, if $y \, F \, x$ then $x \, F \, y$.

(ii) For each $(x, y) \in P \times P$, if $y \, B \, x$ then $x \, B \, y$.

(iii) For each $(x, y) \in P \times P$, if $y \, M \, x$ then $x \, M \, y$.

(b) True or false?

(i) For each $(x, y) \in P \times P$, if $y \, F \, x$ then $x \not{F} y$.

(ii) For each $(x, y) \in P \times P$, if $y \, B \, x$ then $x \not{B} y$.

(iii) For each $(x, y) \in P \times P$, if $y \, M \, x$ then $x \not{M} y$.

(c) Write the denial of each statement given in part (a). [For example, the denial of statement (i) of part (a) is:

There is an $(x, y) \in P \times P$ such that $y \, F \, x$ and $x \not{F} y$]

* * *

Relations such as fatherhood are called <u>asymmetric relations</u>.

> A relation R among the members of a set S is asymmetric if and only if, for each $(x, y) \in S \times S$, if $y \, R \, x$ then $x \not R \, y$.

* * *

4. Which of the relations given in Exercise 8 of Part G are asymmetric?

5. (a) Give a relation which is asymmetric [but different from those in Exercise 8 of Part G].
   (b) Give a relation which is nonsymmetric but not asymmetric.
   (c) Give a relation which is asymmetric but not nonsymmetric.

6. The relations $\geq$ and B [brotherhood] are neither symmetric nor asymmetric. Which of the following statements are true?
   (a) $\forall_x \forall_y$ if $y \geq x$ and $x \geq y$ then $x = y$
   (b) $\forall_{x \in P} \forall_{y \in P}$ if $y \, B \, x$ and $x \, B \, y$ then $x = y$

* * *

Relations such as $\geq$ are called antisymmetric relations.

> R is an antisymmetric relation among the members of a set S if and only if
> $\forall_{x \in S} \forall_{y \in S}$ if $y \, R \, x$ and $x \, R \, y$ then $x = y$.

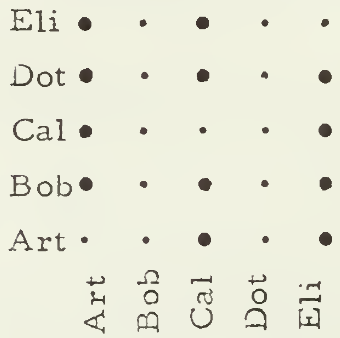
* * *

7. Which of these relations are antisymmetric?
   (a) $\{(x, y) \in L \times L: \ y$ is perpendicular to $x\}$
   (b) $\{(x, y) \in S \times S: \ y \subseteq x\}$ where S is the set of all subsets of $\{1, 2, 3, 4\}$
   (c) $\{(x, y): \ x + 2y = 3\}$ (d) $\{(x, y): \ x^2 = y^2\}$
   (e) $\{(x, y) \in I^+ \times I^+: \ x = \sqrt{y}\}$

8. Describe [list the elements of] a nonempty relation with the least number of elements whose field is $\{1, 2, 3, 4, 5\}$ and which is
   (a) reflexive and symmetric (b) reflexive and nonsymmetric
   (b) irreflexive and symmetric (d) irreflexive and nonsymmetric
   (c) irreflexive and asymmetric (f) reflexive and asymmetric

I. 1. Here is a picture of a relation whose field is the set of all children in some family.

$$C = \{\text{Art, Bob, Cal, Dot, Eli}\}$$



(a) Is this relation reflexive? Symmetric?

(b) What is the domain of the relation? Range?

(c) How many more ordered pairs must be included to get a relation which is reflexive and has the same field? Symmetric?

2. Consider the relation $\{(x, y) \in C \times C\colon$ y is older than x$\}$. Here is a list of the ordered pairs which belong to this relation.

(Bob, Cal), (Bob, Art), (Bob, Eli), (Bob, Dot),
(Cal, Art), (Cal, Eli), (Cal, Dot), (Art, Eli),
(Eli, Dot), (Art, Dot)

(a) Who is older, Cal or Eli?

(b) Who is older, Art or Bob?

(c) Who is the oldest? The youngest?

3. Consider the relation

$$\{(x, y) \in I^+ \times I^+\colon \exists_{m \in I^+}\, x = my\}.$$

Another name for this relation is:

$\{(x, y) \in I^+ \times I^+\colon$ y is a factor of x with respect to $I^+\}$

Let's call this relation '$|$'.

(a) Draw a graph of $|$. What is the domain of $|$? Range?

(b) Is $|$ reflexive? Symmetric?

☆ J. Here is a summary of the definitions of reflexive, irreflexive, symmetric, asymmetric, and antisymmetric relations, as well as definitions of transitive and intransitive relations.

A relation R among the elements of a set M is

reflexive........ $\forall_{x \in \mathfrak{F}_R}$ x R x

irreflexive...... $\forall_{x \in \mathfrak{F}_R}$ x $\not R$ x

symmetric...... $\forall_{x \in M} \forall_{y \in M}$ if y R x then x R y

asymmetric..... $\forall_{x \in M} \forall_{y \in M}$ if y R x then x $\not R$ y

antisymmetric... $\forall_{x \in M} \forall_{y \in M}$ if y R x and x R y then x = y

transitive....... $\forall_{x \in M} \forall_{y \in M} \forall_{z \in M}$ if y R x and z R y then z R x

intransitive ..... $\forall_{x \in M} \forall_{y \in M} \forall_{z \in M}$ if y R x and z R y then z $\not R$ x

For each relation given below list the properties which it has.

1. $\{(x, y) \in P \times P$: y is a sister of x$\}$
2. $\{(x, y) \in P \times P$: y goes to the same school as x$\}$
3. $\{(x, y) \in P \times P$: y lives less than a mile from x$\}$
4. $\{(x, y) \in P \times P$: y does not live on the same street as x$\}$
5. $\{(x, y) \in P \times P$: y is in the same clubs as x$\}$
6. $\{(x, y) \in P \times P$: y sits in the row directly behind x at assembly$\}$
7. $\{(x, y) \in I \times I$: x - y or y - x is an even number$\}$
8. $\{(x, y) \in I \times I$: x and y have a common integral factor $\neq 1\}$
9. $\{(x, y) \in E \times E$: x is equivalent to y$\}$ where E is the set of all equations
10. $\{(x, y) \in A \times A$: y is equivalent to x$\}$ where A is the set of all algebraic expressions in 't' and 'u'
11. $\{(x, y) \in E \times E$: y has the same roots as x$\}$
12. $\{(x, y) \in S \times S$: $y \subseteq x\}$ where S is the set of all subsets of $\{1, 2, 3, 4, 5\}$
13. $\{(x, y) \in I \times I$: x is even and xy = 7$\}$

14. $\{(x, y)\colon\ y = x\}$

15. $\{(x, y)\colon\ y > x\}$

16. $\{(x, y)\colon\ y \leq x\}$

17. $\{(x, y)\colon\ x^2 + y^2 = 1\}$

18. $\{(x, y) \in P \times P\colon$ y is the spouse of x$\}$

19. $\{(x, y) \in P \times P\colon$ y is an ancestor of x$\}$

20. $\{(x, y) \in P \times P\colon$ y is a son of x$\}$

21. $\{(x, y) \in I^+ \times I^+\colon$ y and x have no common factor $\neq 1$ with respect to $I^+\}$

22. $\{(x, y) \in I \times I\colon$ y is a factor of x with respect to $I\}$

23. $\{(x, y) \in L \times L\colon$ y is parallel to x$\}$

24. $\{(x, y) \in L \times L\colon$ y is perpendicular to x$\}$

25. $\{(x, y) \in L \times L\colon$ y is parallel to x or $y = x\}$

☆26. Prove each of the following, or give a counter-example.

(a) For each relation R, if R is symmetric and transitive then R is reflexive.

(b) $\forall_R$ if R is asymmetric then R is irreflexive.

(c) $\forall_R$ if R is irreflexive then R is asymmetric.

(d) $\forall_R$ R is transitive and asymmetric if and only if R is transitive and irreflexive.

(e) $\forall_R$ if R is reflexive then $\widetilde{R}$ [with respect to $\mathfrak{J}_R \times \mathfrak{J}_R$] is irreflexive.

(f) $\forall_R$ if R is symmetric then $\widetilde{R}$ is symmetric.

(g) $\forall_R$ if R is transitive then $\widetilde{R}$ is transitive.

(h) $\forall_R$ if R is asymmetric then R is antisymmetric.

(i) $\forall_R$ if R is antisymmetric then R is asymmetric.

K. 1. Tell which of the sets listed below are functions.

(a) $\{(9, 5), (4, 3), (8, 9), (6, 5), (7, 1)\}$

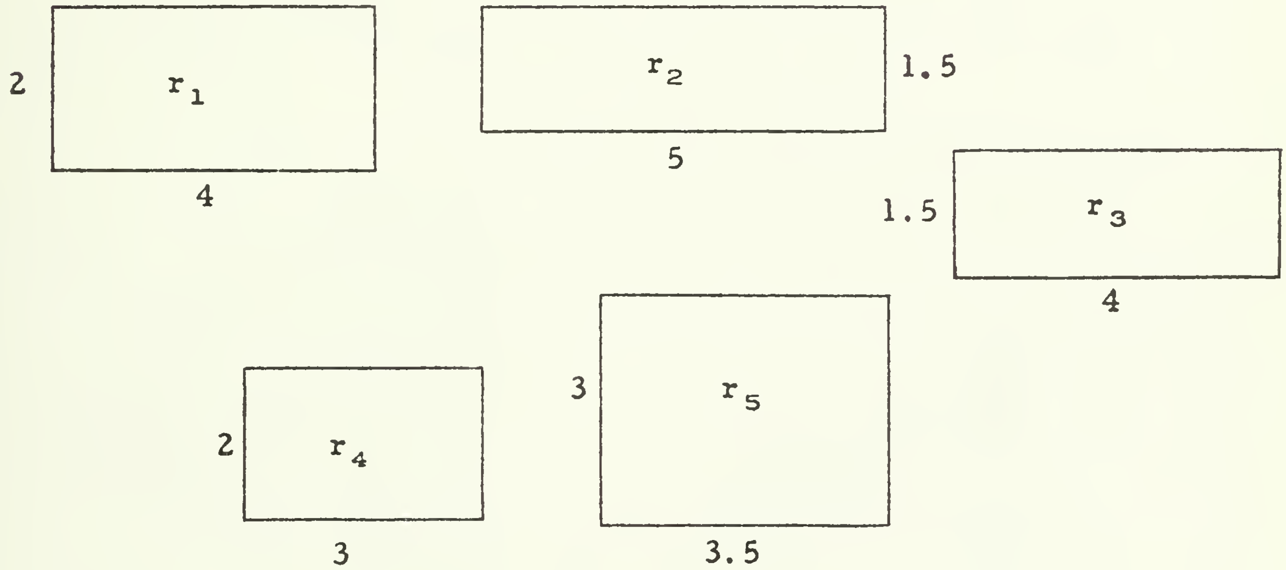
(b) $\{(2, 5), (3, 8), (8, 3), (6, 5), (3, 4)\}$

(c) $\{(5, 17), (82, 61), (0, -3), (4, 2), (-3, 9)\}$

(d) $\{(x, y): x \leq y\}$ (e) $\{(x, y): x \neq y\}$ (f) $\{(x, y): y = (x - 1)^2\}$

2. Which of the relations given in Exercise 1 have converses which are functions?

3. Consider the set S which consists of the five rectangles shown below.



(a) Is $\{(r_1, 4), (r_2, 5), (r_3, 4), (r_4, 3), (r_5, 3.5)\}$ a function?

(b) Is $\{(r_1, 2), (r_2, 1.5), (r_3, 1.5), (r_4, 2), (r_5, 3)\}$ a function?

(c) Consider the set of 5 ordered pairs each of whose first components is one of these five rectangles, and each of whose second components is the corresponding perimeter. Is this set of ordered pairs a function?

(d) Repeat (c) for area-measure.

(e) Consider the set of 5 ordered pairs each of whose first components is the perimeter of one of these five rectangles, and each of whose second components is the corresponding area-measure. Is this a function? Is the converse of this set of ordered pairs a function?

(f) Suppose P is the function whose domain is S and whose range is the set of perimeters of the five rectangles in S. Complete each of the following sentences.

(1) $P(r_1) =$ ________ (2) $P(r_4) =$ ________

(3) $P(r_2) + P(r_3) =$ ________ (4) $\frac{1}{2} P(r_5) =$ ________

4. (a) $f = \{(1, 5), (2, 6), (3, 1), (4, 2), (5, 5), (6, 3)\}$

(1) $f(1) =$ ________ (2) $f(4) =$ ________

(3) $f(6) =$ ________ (4) $f(\_\_\_) = 1$

(5) $f(f(4)) =$ ________ (6) $f(f(6)) =$ ________

(7) $f(3) + f(2) =$ ________ (8) $f(4) \cdot f(2) =$ ________

(9) $\mathcal{D}_f =$ ________________ (10) $\mathcal{R}_f =$ ________________

(b) $g = \{(x, y): y = 5x - 12\}$

(1) $g(2) =$ ________ (2) $g(8) =$ ________

(3) $g(0) =$ ________ (4) $g(-3) =$ ________

(5) $g(4) =$ ________ (6) $g(-1) =$ ________

(7) $g(g(4)) =$ ________ (8) $g(g(g(3))) =$ ________

(9) $g(\_\_\_) = 13$ (10) $g(\_\_\_\_\_\_\_) = 1$

(11) $g(2a) =$ ________ (12) $g(3b + 1) =$ ________

(13) $g(c/5) =$ ________ (14) $g(\frac{d + 12}{5}) =$ ________

(c) $h = \{(x, y): y = 3x - 2\}$ and $j = \{(x, y): y = 6x + 1\}$

(1) $h(4) =$ ________ (2) $h(6) =$ ________

(3) $j(4) =$ ________ (4) $j(6) =$ ________

(5) $h(j(5)) =$ ________ (6) $j(h(-3)) =$ ________

(7) Solve these equations.

(i) $h(a) = 10$ (ii) $j(b) = 49$

(iii) $h(2a) = 24$ (iv) $j(3b - 5) = -11$

(v) $h(k) = k$ (vi) $j(k) = k$

(vii) $h(p) = j(p)$ (viii) $h(2p + 1) = j(p)$

(d) $F = \{(x, y): 3x + 7y - 2 = 0\}$ and $G = \{(x, y): 6x + 5y - 3 = 0\}$

(1) $F(-4) =$ _____ (2) $G(0) =$ _____ (3) $F(0) =$ _____

(4) $G(-2) =$ _____ (5) $F(2a) =$ _____ (6) $G(3k) =$ _____

(7) $F($____$) = 6m$ (8) $G($____$) = 7n - 2$

(9) Solve the equation: $F(p) = G(p)$

(e) $H = \{(x, y): y = x^2 - 9x + 14\}$

(1) $H(3) =$ _____ (2) $H(-2) =$ _____ (3) $H(0) =$ _____

(4) $H(7) =$ _____ (5) $H(2) =$ _____ (6) $H(9/2) =$ _____

(7) $H(2a) =$ _______ (8) $H(5b) =$ _______

(9) $H(2m - 1) =$ _______ (10) $H(3n + 2) =$ _______

(11) $H(p + \frac{9}{2}) =$ _______ (12) $H(2k^2) =$ _______

(f) $M(x) = 4x + 5$, $\mathfrak{D}_M$ = the set of real numbers

(1) $M(3) =$ _______ (2) $M(-2) =$ _______

(3) $M($____$) = 21$ (4) $M($____$) = 0$

(5) $M(k) =$ _______ (6) $M(k + h) =$ _______

(7) $M(a + b) - M(a) =$ _______ (8) $M(5c + d) - M(5c) =$ _____

(9) Solve: $M(x) = x$

(g) $s(t) = 16t - 16t^2$, $\mathfrak{D}_s = \{t: 0 \leq t \leq 1\}$

(1) $s(0) =$ _________ (2) $s(1) =$ _________

(3) $s(\frac{1}{2}) =$ _________ (4) $s(\frac{3}{4}) =$ _________

(5) $s(\frac{1}{4}) =$ _________ (6) $s(-\frac{1}{4}) =$ _________

(7) $s(\frac{5}{2}) =$ _________ (8) $s(\frac{2}{5}) =$ _________

(9) The largest value of s is _____.

☆ (10) Show that if $t_0 \neq t_1$ but $s(t_0) = s(t_1)$ then $t_0 + t_1 = 1$.

L. 1. Graph the function $y = 8 - 2x$.

2. Graph the doubling function.

3. Graph f where

$$f(x) = \begin{cases} 3, & \text{for } x \geq 1 \\ -x, & \text{for } x < 1. \end{cases}$$

4. Graph the function $\frac{7}{3 - x}$.

5. Graph $f(x) = [\![x]\!]$. ['$[\![x]\!]$' is an abbreviation of 'the greatest integer not greater than x'. So, for example, $[\![5.3]\!] = 5$, $[\![5.9]\!] = 5$, $[\![12]\!] = 12$, $[\![1/2]\!] = 0$, and $[\![-5.3]\!] = -6$.]

6. Graph $[\![x + 1]\!]$.

7. Graph $[\![x]\!] + 1$.

8. Graph $x - [\![x]\!]$.

9. Graph $[\![x]\!] - x$.

10. Graph $\{(x, y): [\![y]\!] = [\![x]\!]\}$. Is this a function?

11. The <u>signum</u> <u>function</u>, sg, is defined as follows:

$$sg(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

(a) Graph sg.

(b) Graph $x[sg(x)]$.

(c) Graph $2x[sg(x)]$.

(d) Graph $(2x + 1)[sg(x)]$.

M. 1. Suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{10, 11, 12, 13\}$, and f is the function which maps A on B in such a way that $f(1) = 10$, $f(2) = 11$, $f(3) = f(4) = 12$, and $f(5) = f(6) = 13$. List the ordered pairs in f.

2. Suppose f is a function which maps the set of real numbers on itself in such a way that the image of each real number is its opposite. Write a brace-notation name for f.

3. If f is the doubling function [with $\mathfrak{D}_f = \mathfrak{R}_f$ = the set of real numbers], what is the image of 8? Of what real number is -13 the image?

4. Write a brace-notation name for the mapping f which takes each real number to 1 more than twice its square.

5. Write a brace-notation name for the mapping g which takes each real number x to 3x + 7.

6. The mapping described by '$x \rightarrow 2x + 1$' is the mapping which takes each real number x to the real number $2x + 1$. In other words, it is the function $\{(x, y): y = 2x + 1\}$. Draw a graph of each mapping described below.

(a) $x \rightarrow x + 5$ (b) $x \rightarrow 3x - 2$

(c) $x \rightarrow x^2$ (d) $x \rightarrow (x - 1)^2$

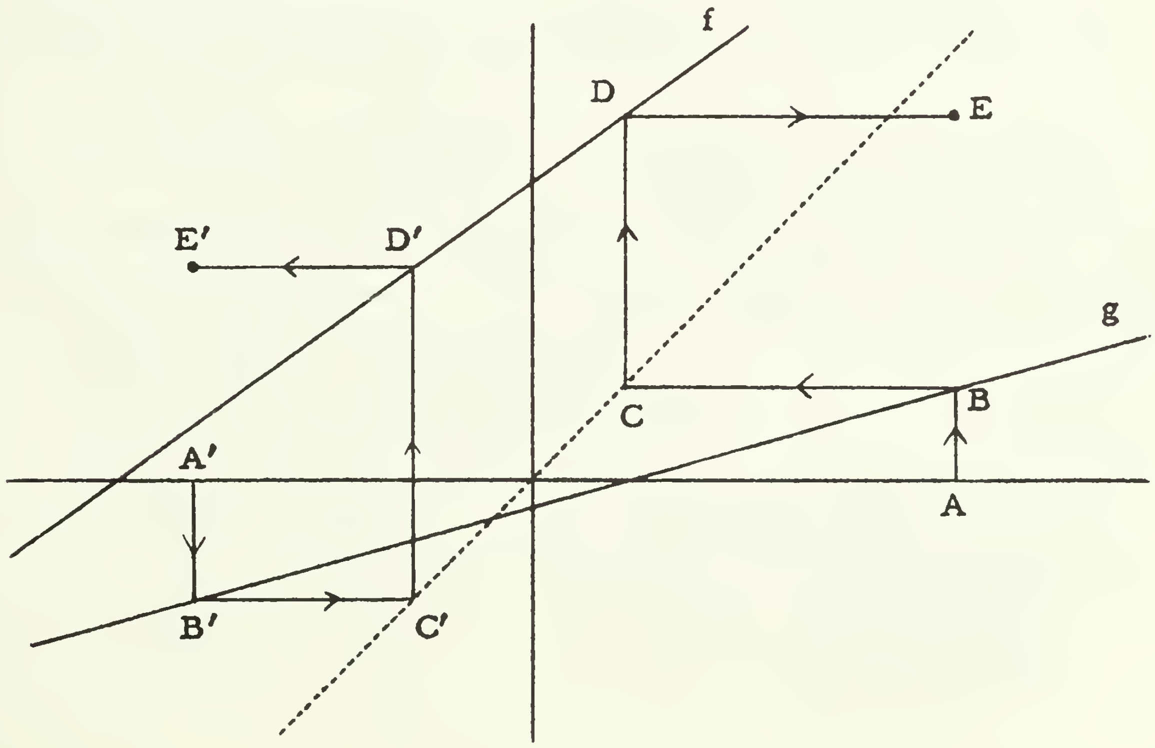
(e) $x \rightarrow |x| + 1$ (f) $x \rightarrow |x - 3|$

7. Use a diagram like those on page 5-64 to picture each mapping described below.

(a) $x \rightarrow \frac{1}{2}x$ (b) $x \rightarrow |x|$

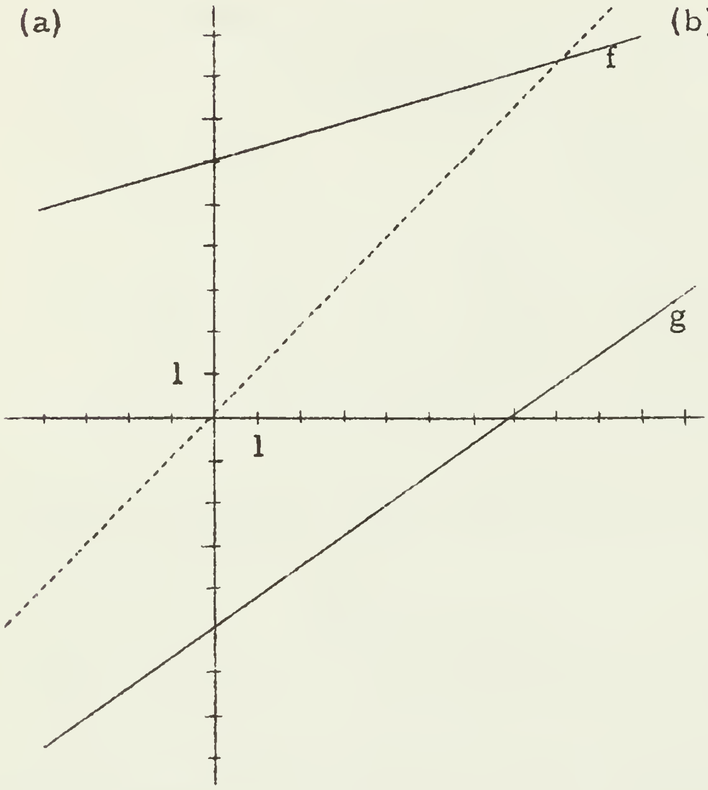
(c) $x \rightarrow 2x - 3$ (d) $x \rightarrow 0$

N. 1. The diagram below shows how to find points in the graph of $f \circ g$ when you have the graphs of f and g. Figure out how it works. [The dashed line is the graph of $\{(x, y): y = x\}$. The points E and E′ are two points on the graph of $f \circ g$. Find some more, and complete the picture.]

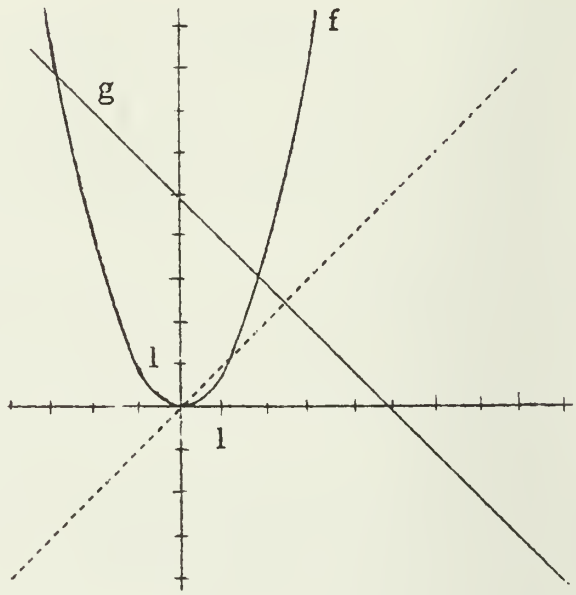


In each of the following exercises, use the graphical method to draw the graph of $f \circ g$.

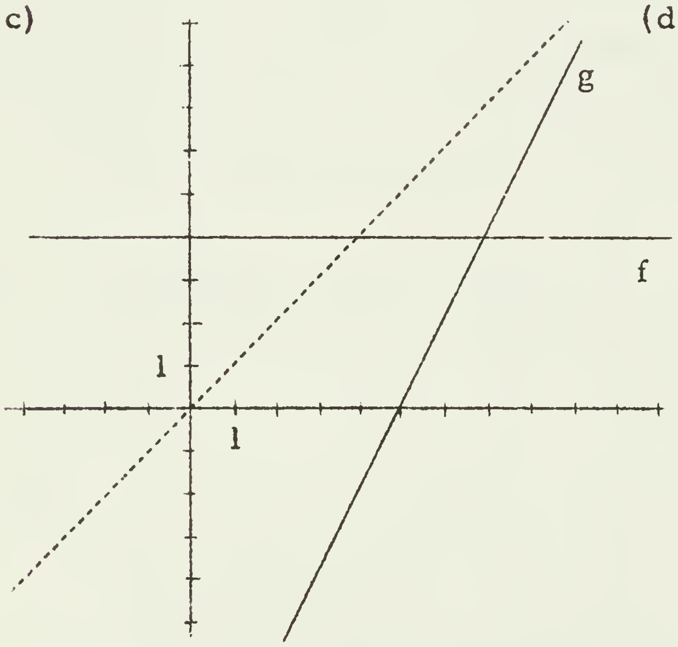
(a)



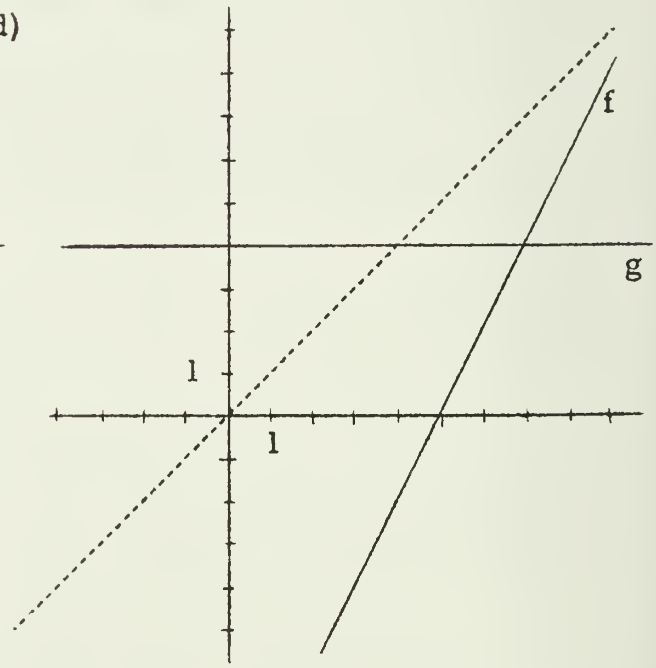
(b)



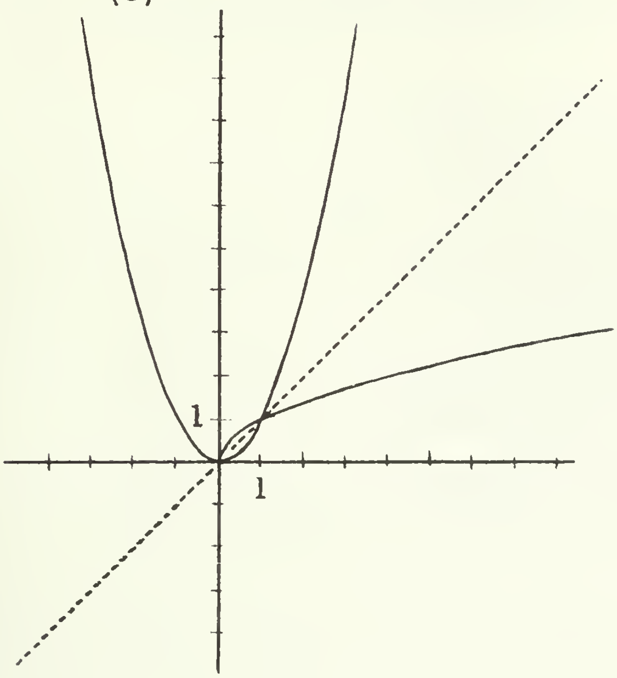
(c)



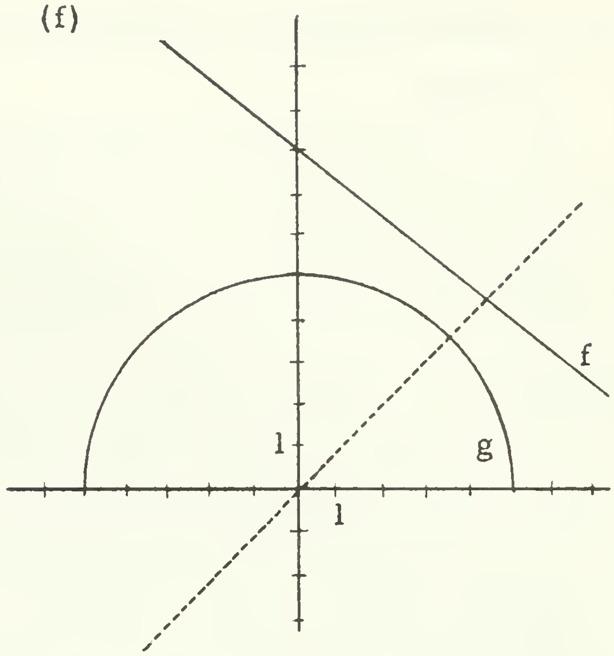
(d)



(e)



(f)



2. Suppose g is the function described by '$x \rightarrow 2x + 3$' and f is the function described by '$x \rightarrow 3x + 5$'. Describe $f \circ g$.

   (a) $g: x \rightarrow 7x + 1$; $f: x \rightarrow x - 5$

   (b) $g: x \rightarrow 2x - 3$; $f: x \rightarrow 2x - 3$

   (c) $g: x \rightarrow 3 - 5x$; $f: x \rightarrow x$

   (d) $g: x \rightarrow \frac{1}{2}x - 3$; $f: x \rightarrow 2x + 6$

   (e) $g: x \rightarrow 8x - 4$; $f: x \rightarrow x^2$

   (f) $g: x \rightarrow 2x - 5$; $f: x \rightarrow 3x^2 - 2x + 5$

<u>O</u>. 1. Write a brace-notation name for each function listed below.

   (a) adding 2 [<u>Answer</u>. $\{(x, y): y = x + 2\}$]

   (b) multiplying by 7 (c) squaring

   (d) the inverse of adding 2

   (e) the inverse of multiplying by 7

   (f) absolute valuing (g) oppositing

   (h) sameing (i) the inverse of oppositing

   (j) doubling (k) the inverse of tripling

2. Tell which of these functions have inverses, and describe the inverse.

(a) $A = \{(4, 7), (8, 9), (12, 11), (16, 13)\}$

(b) $B = \{(5, 6), (6, 7), (8, 9), (10, 5), (11, 6)\}$

(c) $C(x) = x - 1$, $\mathcal{D}_C$ = the set of real numbers

(d) $D(x) = 3x + 5$, $\mathcal{D}_D$ = the set of real numbers

(e) $f(x) = 4x - 5$, $\mathcal{D}_f$ = the set of real numbers

(f) $g(x) = 2x^2$, $\mathcal{D}_g$ = the set of real numbers

(g) $F = \{(x, y): \ y = 3x - 3\}$

(h) $G = \{(x, y): \ y = 4 - 9x\}$

(i) $H = \{(x, y): \ x + 2y = 8\}$

(j) $K = \{(x, y): \ 3x - 5y - 7 = 0\}$

(k) $M = \{(x, y): \ 2(y - 7) + 3(x - 5) = 2(x - 2) + x - 1\}$

3. $f = \{(x, y): \ y = \frac{3x - 2}{5}\}$ and $g = \{(x, y): \ y = \frac{2x - 4}{7}\}$

(a) $f(4) =$ ________ (b) $g(9) =$ ________

(c) $f^{-1}(5) =$ ________ (d) $g^{-1}(4) =$ ________

(e) $f^{-1}(0) =$ ________ (f) $g^{-1}(0) =$ ________

(g) $f^{-1}(f(14)) =$ ________ (h) $f(f^{-1}(14)) =$ ________

(i) $f^{-1}(3a) =$ ________ (j) $g^{-1}(2b) =$ ________

(k) Solve these equations.

(i) $f(a) = a$ (ii) $g(a) = a$

(iii) $f^{-1}(a) = a$ (iv) $g^{-1}(a) = a$

(v) $f(a) = g(a)$ (vi) $f^{-1}(a) = g^{-1}(a)$

(vii) $f(g(a)) = a$ (viii) $g(f(a)) = a$

(ix) $f^{-1}(g^{-1}(a)) = a$ (x) $g^{-1}(f^{-1}(a)) = a$

4. For each of the mappings described below, describe its inverse if it has one.

   (a) $x \to 3x - 6$ [Answer. $x \to \frac{1}{3}x + 2$]

   (b) $x \to 5x + 10$ (c) $x \to 2x - 12$ (d) $x \to x^2 + 1$

   (e) $x \to 7x - 5$ (f) $x \to |x| - 1$ (g) $x \to 4 + 3x$

5. For each of the mappings listed in Exercise 4, find an argument [if there is one] which is its own image. [Sample. (a) The mapping described by '$x \to 3x - 6$' takes 3 to $3 \cdot 3 - 6$, or 3. So, the argument 3 is its own image.]

P. You may recall the number plane games you worked in Unit 4. You were given a rule, for example:

$$(x, y) \to (x + 1, y - 2)$$

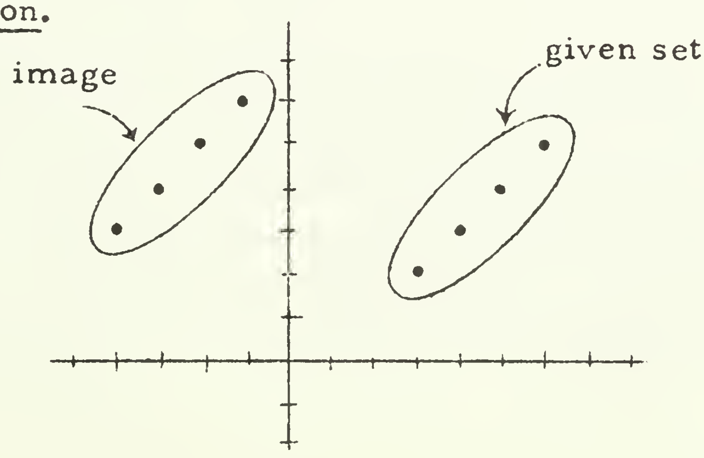
and told to start with a certain point, say, (5, 3), and apply the rule one or more times. Applying the rule once amounts to "jumping" from (5, 3) to (6, 1). Applying it again amounts to jumping from (6, 1) to (7, 4). Etc.

The rule for a number plane game actually defines a function which maps points of the plane onto points of the plane. That is, the domain and range of such a function are subsets of the number plane.

In each of the following exercises, you are given a number-plane-game function and a set in the number plane. Your job is to graph the given set, apply the function to each point in the set, and graph the resulting image.

Sample. Function: $(x, y) \to (x - 7, y + 1)$
Set: $\{(3, 2), (4, 3), (5, 4), (6, 5)\}$

Solution.



1. Function: $(x, y) \to (x + 3, y - 5)$
   Set: $\{(2, 1), (3, 3), (4, 5)\}$

2. Function: $(x, y) \to (x + 3, y - 5)$
   Set: $\{(x, y): y = 2x - 3 \text{ and } 2 \leq x \leq 4\}$

3. Function: $(x, y) \to (x, -y)$
   Set: $\{(4, 4), (5, 3), (6, 2), (7, 1)\}$

4. Function: $(x, y) \to (x, -y)$
   Set: $\{(x, y): y = 2x + 1 \text{ and } 0 \leq x \leq 3\}$

5. Function: $(x, y) \to (x, -y)$
   Set: $\{(x, y): y = 2x + 1\}$

6. Function: $(x, y) \to (-x, y)$
   Set: $\{(x, y): y = x\}$

7. Function: $(x, y) \to (-y, x)$
   Set: $\{(x, y): x = 3 \text{ and } 0 < y < 4\}$

8. Function: $(x, y) \to (-y, x)$
   Set: the union of the segments $\overline{(0, 3)(3, 3)}$ and $\overline{(3, 3)(3, 0)}$

9. Function: $(x, y) \to (-y, x)$
   Set: $\overline{(-3, 0)(-3, 3)} \cup \overline{(-3, 3)(0, 3)}$

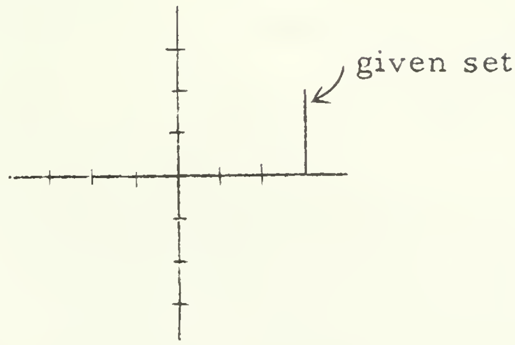
10. Function: $(x, y) \to (-y, x)$
    Set: the square whose vertices are $(3, 3)$, $(-3, 3)$, $(-3, -3)$, and $(3, -3)$

*

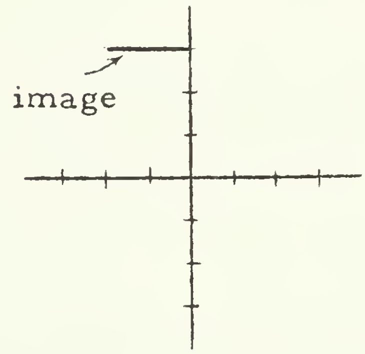
In the preceding exercises you learned something about functions which map points of the number plane onto points of the number plane. You probably discovered easy graphical methods for finding the result of applying a function like the one given by the rule:

$$f: (x, y) \to (-y, x)$$

to a set like the one pictured below.



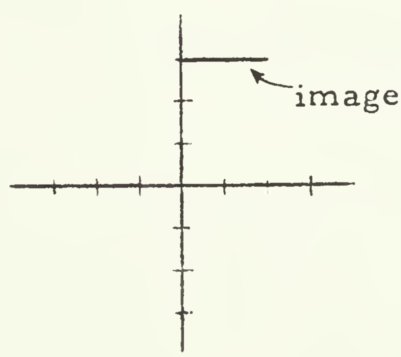
You would get the graph of the image by rotating this picture counter-clockwise a quarter-turn.



If you apply the function given by the rule:

$$g: \ (x, y) \rightarrow (y, x)$$

to the set shown in the first picture, the image is obtained by reflecting the given set in the line $\{(x, y): \ y = x\}$.



*

11. What image do you obtain if you apply the function f to a square S whose center is the origin and whose sides are parallel to the axes?

12. What is the image of S under g?

13. What is the image of S under $f \circ g$?

14. What is the image of S under $g \circ f$?

15. Suppose R is a rectangle whose vertices are (3, 2), (−3, 2), (−3, −2), and (3, −2). What is the image of R under f?

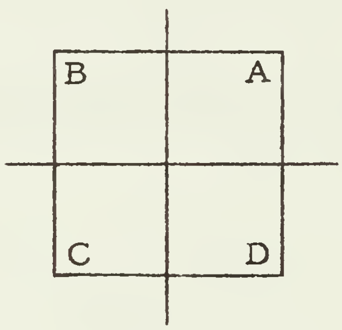
16. Suppose R is the rectangle described in Exercise 15, and T is its image under f. What is the image of R under

    (a) g (b) $f \circ f$ (c) $f \circ g$ (d) $g \circ f$ (e) $g \circ g$

17. In view of your answers to (c) and (d) of Exercise 16, should you conclude that $f \circ g = g \circ f$? [Can you find a point whose image under $f \circ g$ is different from its image under $g \circ f$?]

*

In the preceding exercises you have noticed that there are different functions which map a square onto itself. For example, if ABCD is the square S of Exercise 11, then the function $f[(x, y) \to (-y, x)]$ maps



A on B, B on C, C on D, and D on A. On the other hand, the function $g[(x, y) \to (y, x)]$ maps A on A, B on D, C on C, and D on B. There are other such mappings of S on itself, but not very many.

One way of discovering them is to cut out a cardboard square, and label the corners 'A', 'B', 'C', and 'D' on both sides of the cardboard. Put the cardboard square on a sheet of paper, trace around it, and copy the labels written on the corners onto the paper. Lift up the cardboard and see <u>how many</u> ways you can place it on the square in the drawing.

[Don't forget that one way is to put it right back the way it was. This way is the mapping which takes A onto A, B onto B, C onto C, and D onto D.]

You may find it more interesting to find the total number of ways by doing the following. Ask yourself how many choices there are for the image of A when S is mapped onto itself. Then, for each of these choices, how many choices are there for the image of B. Having chosen images for A and B, how many choices of image are there for C?

One way to keep track of these mappings as you discover them is to fill out a row in the following table for each mapping of the square on itself.

	A	B	C	D
(1)	B	C	D	A
(2)	A	D	C	B
(3)	A	B	C	D
		.		
		.		
		.		

The sample rows filled out in the table correspond with the functions f, g, and the function which leaves each point where it is.

18. Complete this table before reading further.

*

Here is a list of eight number plane mappings which will map a square like S onto itself.

f_1: $(x, y) \to (x, y)$ f_5: $(x, y) \to (y, x)$

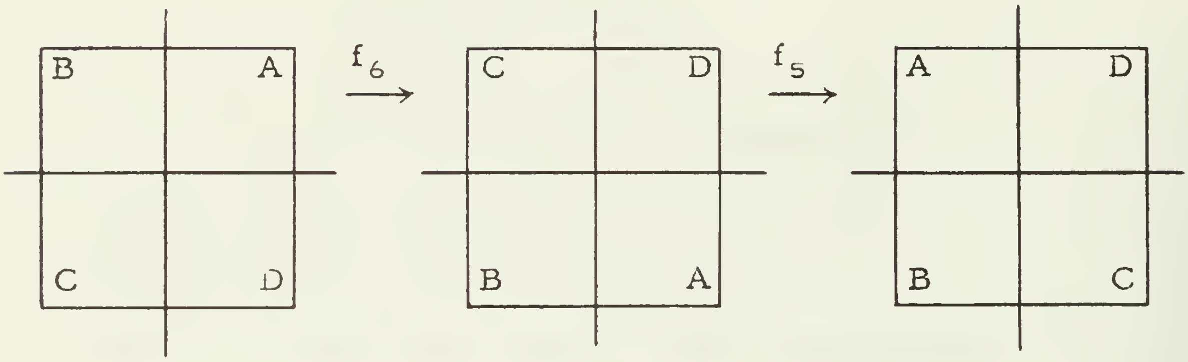
f_2: $(x, y) \to (-y, x)$ f_6: $(x, y) \to (x, -y)$

f_3: $(x, y) \to (-x, -y)$ f_7: $(x, y) \to (-y, -x)$

f_4: $(x, y) \to (y, -x)$ f_8: $(x, y) \to (-x, y)$

Notice that f_1 is the mapping that doesn't "move" anything, that f_2 and f_5 are the mappings f and g referred to earlier, that $f_3 = f \circ f$, that $f_6 = g \circ f$, and that $f_7 = g \circ f \circ f$. What is f_4? f_8?

Suppose we compose one of these mappings with another, say, f_5 with f_6. This gives us the mapping $f_5 \circ f_6$. Let's see it in action.



So, $f_5 \circ f_6$ is the mapping that rotates S counterclockwise a quarter-turn. In other words, $f_5 \circ f_6 = f_2$.

19. Complete the following composition table. [It may help to notice that $f \circ g = g \circ f \circ f \circ f$.]

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1								
f_2								
f_3								
f_4								
f_5						f_2		
f_6								
f_7								
f_8								

20. Which of the eight mappings are rotations? Which are reflections?

21. Which of the eight mappings is f_2^{-1}? f_6^{-1}? $(f_2 \circ f_6)^{-1}$? $f_6^{-1} \circ f_2^{-1}$?

Q. 1. For each exercise, tell whether h is a function of g, and if it is, give a function f such that $h = f \circ g$.

(a) $g = \{(x, y): y = 7x + 1\}$ and $h = \{(x, y): y = 7x - 4\}$

(b) $g = \{(x, y): y = 2 - 3x\}$ and $h = \{(x, y): y = 3x - 2\}$

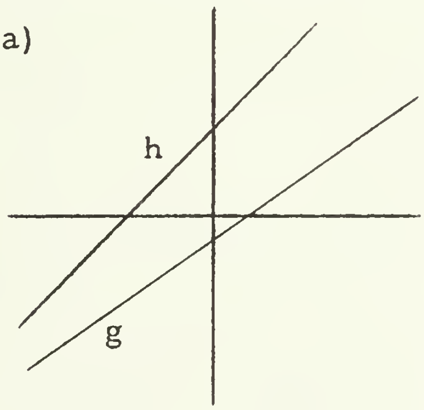
(c) $g = \{(x, y), x \geq 0: y = 2 - 3x\}$ and $h = \{(x, y), x \geq 0: y = 3x - 2\}$

(d) $g = \{(x, y), x < 0: y = 2 - 3x\}$ and $h = \{(x, y), x \geq 0: y = 3x - 2\}$

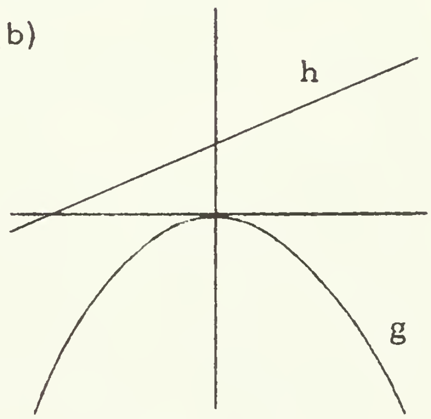
(e) $g = \{(x, y): y = x^2 - x + 1\}$ and $h = \{(x, y): y = 3x^2 - 3x - 5\}$

2. For each exercise, tell whether h is a function of g.

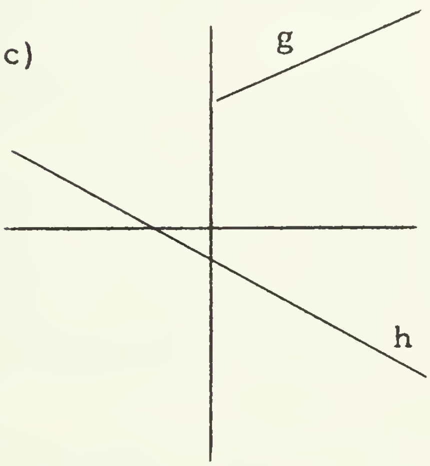
(a)



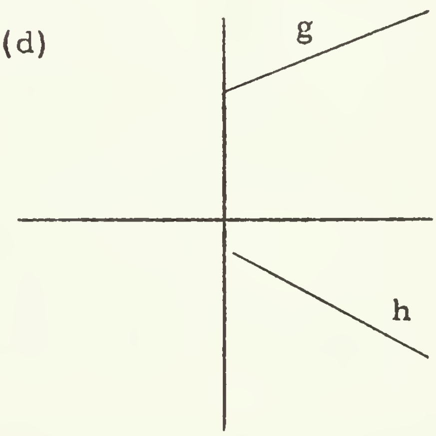
(b)



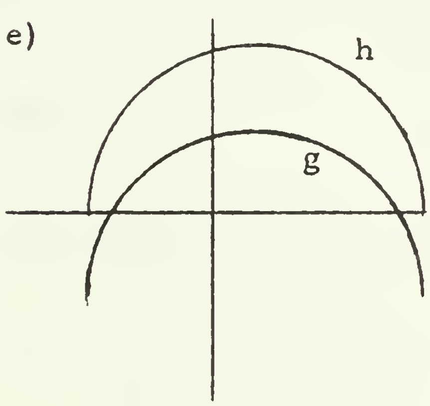
(c)



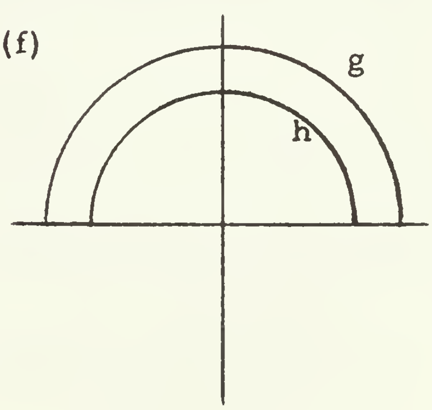
(d)



(e)



(f)



R. 1. Consider the variable quantities M and N where

$$M = \{(\text{Don}, 6), (\text{Mike}, 9), (\text{Sally}, 4)\}$$

and $$N = \{(\text{Don}, 5), (\text{Mike}, 11), (\text{Sally}, 7)\}$$

(a) $M + N = \{$____$\}$

(b) $MN = \{$____$\}$

(c) $2M + N = \{$____$\}$

(d) $M + 2N = \{$____$\}$

(e) $M - N = \{$____$\}$

(f) $(M + N)(M - N) = \{$________________________________$\}$

(g) $M^2 - N^2 = \{$____$\}$

2. Suppose $f(x) = 3x - 5$ and $g(x) = 7x + 3$.

(a) $[f + g](3) =$ ________ (b) $[f + g](-3) =$ ________

(c) $[f - g](5) =$ ________ (d) $[f - g]($____$) = 0$

(e) $[f \circ g](4) =$ ________ (f) $[fg](4) =$ ________

(g) $[g \circ f](4) =$ ________ (h) $[gf](4) =$ ________

(i) $[f \circ g](x) =$ ________ (j) $[fg](x) =$ ________

(k) $[f^2](x) =$ ________ (ℓ) $[g^2](x) =$ ________

(m) $[f^2 + g^2](x) =$ ________ (n) $[f^2 - g^2](x) =$ ________

3. Suppose $A = \{(0, 5), (1, 10), (2, 15)\}$ and $B = \{(0, 8), (1, 13), (2, 18)\}$.

(a) Is B a function of A?

(b) If $h = \{(x, y): y = x + 3\}$, does $B = h \circ A$?

(c) If $g = \{(3, 4), (5, 8), (6, 7), (9, 11), (10, 13), (15, 18)\}$, does $B = g \circ A$?

(d) If $f = \{(5, 8), (6, 9), (10, 12), (12, 14), (15, 18)\}$, does $B = f \circ A$?

(e) If $k = \{(x, y), x \geq 5: y = x + 3\}$, does $B = k \circ A$?

(f) If $m = \{(x, y), x > 5: y = x + 3\}$, does $B = m \circ A$?

S. 1. Each exercise gives the slope and intercept of a linear function. Draw its graph and write its defining equation.

(a) intercept, 1; slope, 1
(b) intercept, 1; slope, -1
(c) intercept, 2; slope, $\frac{1}{2}$
(d) intercept, -3; slope, $\frac{7}{6}$

2. The solution set in (x, y) of each of the following equations is a linear function. Which of these functions have the same slope?

(a) $y = 2x + 5$
(b) $y + \frac{1}{2}x + 5 = 0$
(c) $5(y + 2x) = 1$
(d) $2y - x = 7$
(e) $2(y + 1) = x$
(f) $3y + 6x = 7$
(g) $2(y + x) + 2y = 2$
(h) $\frac{4y + 1}{2} = x$
(i) $7y - 14x + 2 = 0$
(j) $8y + 4x - 1 = 0$
(k) $22x + 11y = -3$
(ℓ) $18x - 9y = 2$
(m) $6y - 3x + 5 = 0$
(n) $3y = \frac{3}{2}x + 5$
(o) $2y = 1 - 4x$
(p) $6y + 3x - 5 = 0$
(q) $-5x - 10y + 2 = 0$
(r) $10y = -5(2 - x)$
(s) $x = \frac{y - 4}{-2}$
(t) $x = \frac{3y - 5}{6}$

3. Find the linear function [if there is one] which contains the two ordered pairs.

(a) (4, 2), (3, -5)
(b) (0, 0), (8, -5)
(c) (6, -1), (-1, 6)
(d) (2, 5), (-3, -8)

4. Find the ordered pair (x, y) [if there is one] which satisfies both equations.

(a) $y = 3x + 5$
$y = 5x + 3$

(b) $y = 6x - 2$
$y = x + 8$

(c) $y = 4x - 1$
$y = 6x - 11$

(d) $x = 3y + 4$
$x = 2y + 5$

(e) $y - x = 4$
$y + x = 8$

(f) $y - 2x = 5$
$y + 2x = 17$

(g) $y - 3x = 5$
$y + 2x = 30$

(h) $2y + x = 10$
$2y - x = 2$

(i) $2y + 5x = 34$
$2y - 4x = -20$

(j) $3y + 2x = 9$
$4y - 5x = 11$

5. Each exercise describes a linear function. Find the linear function which fits the description.

<u>Sample.</u> It contains $(5, -3)$, and its slope is -2.

<u>Solution.</u>
$$\begin{aligned} y &= ax + b \\ -3 &= a5 + b \\ -3 &= -2 \cdot 5 + b \\ 7 &= b \end{aligned}$$

So, the linear function is $\{(x, y): y = -2x + 7\}$.

(a) It contains $(2, 1)$, and its slope is 4.

(b) Its intersection with $\{(x, y): y = 5x - 3\}$ is $\emptyset$, and it contains $(2, 14)$.

(c) Its intersection with $\{(x, y): 6x - 2y + 5 = 0\}$ is $\emptyset$, and its intercept is 9.

(d) For each member of its domain, the corresponding value is twice the corresponding value of $\{(x, y): y = 7x - 1\}$.

(e) $\{(3, 7), (5, 8), (7, 9)\}$ is one of its subsets.

(f) Its intersection with the x-axis is $\{(8, 0)\}$, and its intersection with the y-axis is $\{(0, -2)\}$.

(g) It contains the midpoints of the intervals $\overline{(0, 3), (3, 0)}$ and $\overline{(0, 5), (5, 0)}$.

(h) Its graph is perpendicular to the graph of $\{(x, y): y = 2x\}$ and crosses it at the graph of the origin.

6. Suppose f and g are linear functions and that $f(x) = ax + b$ and $g(x) = cx + d$.

(a) $[f \circ g](x) =$ ________

(b) $[g \circ f](x) =$ ________

(c) $[fg](x) =$ ________

(d) $[gf](x) =$ ________

7. Each table lists ordered pairs which belong to a linear function. Complete the tables.

(a)

3	8
4	11
5	
6	17
	29
15	

(b)

5	7
7	15
7.5	
8	
8.3	
10	

(c)

2	−3
8	−1
9	
13	
−5	
−7	

T. 1. If P is directly proportional to C, and P has the value 35 when C has the value 7, what is the value of P when C has the value 25?

2. If x varies directly as y and x = 12 when y = 4, what is y when x = 5?

3. If x is directly proportional to y and x is 9 when y is 5, what value of y corresponds to the value 15 of x?

4. If 12 hats cost \$60, how much will 17 hats of the same kind cost?

5. If A varies directly as B then ____________________.

(a) AB is a constant variable quantity

(b) A + B is a constant variable quantity

(c) A/B is a constant variable quantity

6. Solve these proportions.

(a) $\frac{y}{3} = \frac{5}{20}$ (b) $\frac{5}{4} = \frac{k}{16}$ (c) $\frac{z}{9} = \frac{14}{36}$

(d) $\frac{x}{80} = \frac{36}{20}$ (e) $\frac{25}{x} = \frac{5}{7}$ (f) $\frac{3}{a} = \frac{1}{24}$

(g) $\frac{3}{4} = \frac{x}{12}$ (h) $\frac{5}{3} = \frac{25}{y}$ (i) $\frac{1}{3} = \frac{k}{k+8}$

(j) $\frac{m+10}{m} = \frac{9}{6}$ (k) $\frac{y}{12-y} = \frac{1}{3}$ (ℓ) $\frac{2}{1} = \frac{21-b}{b}$

7. If 3 pears cost 17 cents, what is the cost of a dozen pears at the same rate?

8. A 3-pound cake requires $1\frac{2}{3}$ cups of sugar. How many cups should be used for a 5-pound cake of the same type?

9. The ratio of cement to sand in a concrete mixture is 1 : 4. How many shovelfuls of cement should be used with 50 shovelfuls of sand?

10. The ratio of the length of a rectangle to its width is 10 : 7. If the rectangle is 35 feet long, what is its width?

11. Which of these formulas expresses the fact that one variable quantity varies directly as another?

    (a) $A + B = 100$ (b) $10C = D$ (c) $\frac{M}{N} = 4$ (d) $PQ = 10$

12. If x varies inversely as y, and x is 12 when y is 4, what is y when x is 6?

13. AB is a constant variable quantity. If $A_1 = 8$, $B_1 = 9$, and $B_2 = 18$, what is A_2?

14. U varies inversely as T and U is 80 when T = 0.25. What value of U corresponds with the value 6 of T?

15. If M is inversely proportional to N and $M_1 = 5 = N_1$, what is the factor of inverse proportionality?

16. Write a formula to express one of the variable quantities in terms of the other. In each case, use 'k' to name the constant variable quantity whose value is the factor of variation.

    (a) The circumference (C) of a circle varies directly as the radius (r).

    (b) The side-measure (s) of an equilateral triangle is directly proportional to the perimeter (P).

    (c) The resistance (R) of a piece of copper wire varies directly as its length (ℓ).

    (d) The volume (V) of a gas under constant pressure varies directly as its absolute temperature (T).

    (e) The resistance (R) of a piece of copper wire varies inversely as the area-measure (A) of a cross-section of the wire.

    (f) The number (n) of items that can be bought for a fixed amount of money is inversely proportional to the cost (c) of each item.

(g) The illumination (I) from a source of light varies inversely as the square of the distance (d) from the source.

(h) The number (f) of vibrations made per second by a violin string of a certain diameter is inversely proportional to the length-measure (ℓ) of the string and directly proportional to the square root of the force (F) with which it is stretched.

17. If $A = \pi r^2$, what is the effect on r when the area-measure is multiplied by 25?

18. If a body falls from rest, the distance fallen from the starting point during an interval of elapsed time varies directly as the square of the elapsed time. If a body falls 402.5 feet in 5 seconds, how many feet does it fall in 10 seconds? How many feet does it fall <u>during</u> the eleventh second.

19. A is directly proportional to B and inversely proportional to C. If the value of A is 9 when the value of B is 18 and the value of C is 30, what is the value of A when the value of B is 42 and the value of C is 25?

20. The area-measure of the surface of a cube varies directly as the square of the measure of its edge. If the area of the surface of a cube whose edge is 5/3 feet long is 50/3 square feet, how long is the edge of a cube whose surface area is 243/8 square feet?

21. The volume-measure (V) of a sphere varies jointly as its diameter (d) and the area-measure (S) of its surface. If $V_1 = a$, $d_1 = b$, $S_1 = c$, $V_2 = p$, and $S_2 = q$, what is d_2?

☆ 22. Suppose A = B + C and B varies directly as D^3 and C varies inversely as D. If $A_1 = -53$, $D_1 = -3$, $A_2 = 14.5$, $D_2 = 2$, and $D_3 = 2$, what is A_3?

☆ 23. The volume of a circular disk varies jointly as its thickness and the square of the radius of its face. Two disks which are 5 centimeters and 7 centimeters thick whose faces have radii 60 centimeters and 30 centimeters, respectively, are melted and formed into 50 congruent disks, each 3 centimeters thick. Find the radius of the face of each of the new disks.

U. 1. Transform each of the following quadratic expressions in 'x' by completing the square, and simplifying.

(a) $x^2 + 7x - 5$ $[(x + \frac{7}{2})^2 - \frac{69}{4}]$ (b) $x^2 + 8x - 5$

(c) $x^2 + 9x - 2$ (d) $x^2 - 10x + 1$

(e) $x^2 - 3x + 5$ (f) $x^2 + 2x + 17$

(g) $4x - 7 + x^2$ (h) $2 - 5x + x^2$

(i) $3x^2 - 6x + 9$ (j) $12x^2 - 2x + 5$

2. Find the equation of the axis of symmetry for each quadratic function.

(a) $f(x) = x^2 + 7x - 5$ (b) $f(x) = x^2 + 8x - 5$

(c) $f(x) = 3 - 2x + x^2$ (d) $f(x) = 6 + 5x - x^2$

(e) $f(x) = 2x^2 + x - 7$ (f) $f(x) = -5x^2 + 7x - 3$

3. The graph of the relation $\{(x, y): x^2 + y^2 = 25\}$ is a circle whose center is the graph of (0, 0) and whose radius is 5. The graphs of the relations listed below are circles. For each relation, give the point whose graph is the center of the circle, and give the radius of the circle.

(a) $\{(x, y): x^2 + y^2 = 36\}$ (b) $\{(x, y): x^2 + y^2 = 2\}$

(c) $\{(x, y): (x - 2)^2 + y^2 = 25\}$ (d) $\{(x, y): (x - 6)^2 + y^2 = 25\}$

(e) $\{(x, y): (x + 5)^2 + y^2 = 25\}$ (f) $\{(x, y): x^2 + (y - 1)^2 = 25\}$

(g) $\{(x, y): x^2 + (y + 7)^2 = 25\}$ (h) $\{(x, y): (x - 2)^2 + (y - 3)^2 = 25\}$

(i) $\{(x, y): (x + 4)^2 + (y - 9)^2 = 25\}$

(j) $\{(x, y): x^2 + 8x + 16 + y^2 - 18y + 81 = 25\}$

(k) $\{(x, y): x^2 - 6x + 9 + y^2 + 8y + 16 = 25\}$

(ℓ) $\{(x, y): x^2 - 6x \qquad + y^2 + 8y \qquad = 0\}$

(m) $\{(x, y): x^2 - 12x \qquad + y^2 - 4y \qquad = -15\}$

(n) $\{(x, y): x^2 + y^2 - 14x - 2y + 14 = 0\}$

(o) $\{(x, y): x^2 + y^2 = 2(10 - 5x - 2y)\}$

V. 1. Solve these equations by transforming to standard form and searching for factors.

(a) $x^2 = 5x - 4$

[Solution. $x^2 - 5x + 4 = 0$; $(x - 4)(x - 1) = 0$; $x = 4$ or $x = 1$; the roots are 4 and 1.]

(b) $x^2 + 10 = 7x$ (c) $a^2 + 3(a + 2) + 3a + 2 = 0$

(d) $m^2 + 15 = -8m$ (e) $6 + x = x^2$ (f) $y^2 = 12 + y$

(g) $p^2 = -7p$ (h) $10 - 9y = -2y^2$ (i) $1 = \frac{1}{2}k^2 - \frac{7}{6}k$

(j) $2x^2 - 3x + 1 = 0$ (k) $2x^2 = 5x - 2$

2. Solve these equations by transforming to standard form and completing the square.

(a) $x^2 + 2x - 4 = 0$ (b) $y^2 + 4y - 21 = 0$ (c) $k^2 - 6k = 16$

(d) $x^2 = 4(2x - 3)$ (e) $x(x + 3) + 2 = 0$ (f) $6 = y(5 + y)$

(g) $m^2 = 2m + 5$ (h) $4(1 - m) = 2m(1 + m) - m^2$

3. Solve by using the quadratic formula.

(a) $m^2 + 3m - 40 = 0$ (b) $3k^2 + 5k = -2$ (c) $c^2 - 10c = -15$

(d) $x = -2(3 - x^2)$ (e) $2(1 + 2r) = 5r^2$ (f) $2x^2 - 4 = 5x$

(g) $\frac{2}{y^2} + 1 = \frac{7}{2y}$ (h) $\frac{1}{x - 1} - \frac{3}{x - 2} + 2 = 0$

4. Find rational approximations [correct to the nearest tenth] to the roots of these equations.

(a) $x^2 + 4x - 2 = 0$ (b) $5k + 4 = 3k^2$

(c) $3(x + 1) = x^2$ (d) $3 = 2y(y + 2)$

5. Find the ordered pairs in these intersections.

(a) $\{(x, y): y = x^2 + 5x - 4\} \cap$ the y-axis

(b) $\{(x, y): y = x^2 + 5x - 4\} \cap$ the x-axis

(c) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 1\}$

(d) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 2\}$

(e) $\{(x, y): y = x^2 + 5x - 4\} \cap \{(x, y): y = 10\}$

6. (a) If one root of the quadratic equation '$x^2 - 5x + k = 0$' is 2, what is the other root?

   (b) For what value of 'n' is '$x^2 - 7x + n = 0$' satisfied by 2?

   (c) What is the sum of the roots of the equation '$x^2 - 3x - 40 = 0$'?

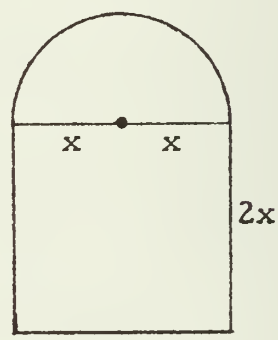
   (d) What is the product of the roots of the quadratic equation '$x^2 - 9x + 18 = 0$'?

   (e) Write a quadratic equation whose roots are 2 and -5.

7. (a) John is 5 years older than Bob. The product of their ages [that is, the product of the number of years in their ages] is 176. Find their ages.

   (b) The length of a rectangle is 3 inches more than its width. If its area is 65 square inches, what is its inch-perimeter?

   (c) The figure shows a square surmounted by a semicircle. If one uses 22/7 as an approximation for π, he obtains 273 as an approximation for the total area. What are the dimensions of the square?



   (d) One number exceeds another by 3. The sum of their squares is 149. What are the numbers?

   (e) One number exceeds another by 5. Their product is 104. What are the numbers?

   (f) A man walks for x hours at x miles per hour. If he had increased his speed by 2 miles per hour, he would have walked 11.25 miles. How far did he actually walk?

   (g) A small orchard has 60 trees and yields, on the average, 400 apples per tree. For each additional tree planted in this orchard, the average yield per tree is reduced by approximately 6 apples. How many trees will give the largest crop of apples for this orchard?

W. 1. Solve these systems of equations.

(a) $\left.\begin{array}{l} a + b = 12 \\ a - b = 4 \end{array}\right\}$

(b) $\left.\begin{array}{l} x + y = 13 \\ x - y = 5 \end{array}\right\}$

(c) $\left.\begin{array}{l} m + n = -6 \\ m - n = -10 \end{array}\right\}$

(d) $\left.\begin{array}{l} 3a + b = 16 \\ 2a + b = 11 \end{array}\right\}$

(e) $\left.\begin{array}{l} x + y = 0 \\ y - x = 6 \end{array}\right\}$

(f) $\left.\begin{array}{l} 4x + 3y = 29 \\ 2x - 3y = 1 \end{array}\right\}$

(g) $\left.\begin{array}{l} 3p + 7q = -4 \\ 2p + 5q = -3 \end{array}\right\}$

(h) $\left.\begin{array}{l} 3x = 4 - 7y \\ 4x = 3y - 7 \end{array}\right\}$

(i) $\left.\begin{array}{l} 5u = 4v \\ \frac{1}{2}u = 2(6 - v) \end{array}\right\}$

(j) $\left.\begin{array}{l} y = 3x \\ x - y = 2 \end{array}\right\}$

(k) $\left.\begin{array}{l} 5a + b + 2 = 0 \\ a + 2b = 5 \end{array}\right\}$

(ℓ) $\left.\begin{array}{l} 3m = p + 1 \\ 3p + 7 = 5m \end{array}\right\}$

(m) $\left.\begin{array}{l} 2x - y = 5 \\ 2x + 3y = 1 \end{array}\right\}$

(n) $\left.\begin{array}{l} x + 2y = 7 \\ 4x - y = 1 \end{array}\right\}$

(o) $\left.\begin{array}{l} 5x + 4y = 27 \\ x - 2y = 11 \end{array}\right\}$

(p) $\left.\begin{array}{l} x + y = 3 \\ 2x - y = 9 \end{array}\right\}$

(q) $\left.\begin{array}{l} 3x - 4y = 8 \\ x + 2y = 1 \end{array}\right\}$

(r) $\left.\begin{array}{l} 3x - y = 4 \\ 2x + 3y = 21 \end{array}\right\}$

(s) $\left.\begin{array}{l} 5x + 3y = 3 \\ 2x + 3y = 12 \end{array}\right\}$

(t) $\left.\begin{array}{l} 4x + y = 5 \\ 2x + y = 9 \end{array}\right\}$

2. Solve these systems of equations.

(a) $\left.\begin{array}{l} x - 2y + 20z = 1 \\ 3x + y - 4z = 2 \\ 2x + y - 8z = 3 \end{array}\right\}$

(b) $\left.\begin{array}{l} 2x - y + z = 5 \\ 4x - 3y = 5 \\ 6x + 2y + 2z = 7 \end{array}\right\}$

(c) $\left.\begin{array}{l} 3x + 2y = 5 \\ 4x - 3z = 7 \\ 6y - 6z = -5 \end{array}\right\}$

(d) $\left.\begin{array}{l} 3x - 2y + z = 4 \\ 2x + 4y - 3z = 9 \\ -x + 8y - 2z = 4 \end{array}\right\}$

(e) $\left.\begin{array}{l} 3x - 2y - 3z = -1 \\ 6x + y + 2z = 7 \\ 9x + 3y + 4z = 9 \end{array}\right\}$

(f) $\left.\begin{array}{l} 3x + 3y - 2z = 0 \\ 4x - 9y + 4z = 6 \\ 5x - 6y + 6z = 6 \end{array}\right\}$

3. Solve these systems of equations.

Sample. $\left.\begin{array}{l} x^2 - xy + y = 5 \\ 2x + y = 3 \end{array}\right\}$

Solution.

$$y = 3 - 2x$$
$$x^2 - x(3 - 2x) + (3 - 2x) = 5$$
$$x^2 - 3x + 2x^2 + 3 - 2x = 5$$
$$3x^2 - 5x - 2 = 0$$
$$(3x + 1)(x - 2) = 0$$
$$x = -\frac{1}{3} \quad \text{or} \quad x = 2$$

$$\begin{array}{l|l} y = 3 - 2\cdot -\frac{1}{3} & y = 3 - 2\cdot 2 \\ \quad = 3 + \frac{2}{3} & \quad = 3 - 4 \\ \quad = \frac{11}{3} & \quad = -1 \end{array}$$

Check. $(-\frac{1}{3})^2 - -\frac{1}{3}\cdot\frac{11}{3} + \frac{11}{3} \stackrel{?}{=} 5$

$$\frac{1}{9} + \frac{11}{9} + \frac{11}{3} \stackrel{?}{=} 5$$
$$\frac{45}{9} = 5 \quad \checkmark$$

$$2^2 - 2\cdot -1 + -1 \stackrel{?}{=} 5$$
$$4 + 2 + -1 = 5 \quad \checkmark$$

The solution set is $\{(-\frac{1}{3}, \frac{11}{3}), (2, -1)\}$.

(a) $\left.\begin{array}{l} x^2 - 3y^2 = 6 \\ x + 2y = -1 \end{array}\right\}$

(b) $\left.\begin{array}{l} y = 2x - 3 \\ 3x^2 = 4 + xy \end{array}\right\}$

(c) $\left.\begin{array}{l} x^2 + y^2 - 3y = 4 \\ 2x - y = 1 \end{array}\right\}$

(d) $\left.\begin{array}{l} x^2 - xy + y^2 = 63 \\ y - x = 3 \end{array}\right\}$

(e) $\left.\begin{array}{l} x^2 - xy - x = 8 \\ y + x = -1 \end{array}\right\}$

(f) $\left.\begin{array}{l} x^2 + xy = 6 \\ 3x - y = 2 \end{array}\right\}$

(g) $\left.\begin{array}{l} 2x^2 - y^2 = 7 \\ 2x + y = 5 \end{array}\right\}$

(h) $\left.\begin{array}{l} x^2 + y^2 - 10y = 24 \\ y = x - 2 \end{array}\right\}$

4. (a) A man bought 3 cans of corn and 2 cans of peas for a total of 69 cents. 2 cans of corn and 3 cans of peas would have cost a total of 71 cents. What would 5 cans of corn and 5 cans of peas cost?

   (b) A razor blade dispenser containing 25 razor blades costs 63 cents. The same dispenser with 50 razor blades costs $1.13. What is the cost of the dispenser by itself?

   (c) A truck with a full load of coal weighs 8000 pounds. The same truck with a third of a load of coal weighs 6600 pounds. How much does the truck weigh?

   (d) At 9 a.m., a man starts on a trip by car; he averages 30 miles per hour. His son starts after him at 11 a.m., and averages 40 miles per hour. When will the son catch up with his father?

   (e) A furniture manufacturer makes tables. The cost of the material for the table top is computed on the basis of the area, while the cost of the material for the legs is computed on the basis of length. A table whose top measures 4 feet by 4 feet and whose legs are 3 feet long costs $42 for materials, while a table 5 feet by 5 feet whose legs are also 3 feet long, made of the same materials, costs $72. What is the cost of a square foot of table top? What is the cost of a foot of material for the legs?

   (f) How many pounds of pure salt must be added to 50 pounds of a 25% solution of salt and water in order to yield a mixture that will be 3% salt?

   (g) A stack containing 8 copies of Unit 1 and 3 copies of Unit 2 is 12.5 inches high. Another stack containing 4 copies of Unit 1 and 5 copies of Unit 2 is 11.5 inches high. How thick is a copy of Unit 1?

   (h) A man gets different rates of pay for work done during his regular hours and for work done overtime. He gets $155 for 35 regular hours and 10 hours overtime. For 30 hours regular time and 15 hours overtime he gets $165. What is his regular hourly rate of pay? What is his overtime rate of pay?

(i) Children paid 25 cents each and adults paid 60 cents each for admission to a movie. A total of \$11.00 was collected. If children had paid 30 cents each and adults 75 cents each, the total collected would have been \$13.50. How many children attended? How many adults?

(j) The intersection of two sets is the empty set. One contains 5 elements more than the other, and their union contains 39 elements. How many elements are there in each set?

(k) A man walks for 3 hours and then rides for 2 hours. He covers a total distance of 84 miles. If he were to walk for 2 hours and ride for 3 hours at the same average rates, he would cover 116 miles. What is his average rate of walking? What is his average rate of riding?

(ℓ) Two machines produce bottle caps. The older machine works for 3 hours and the newer machine for 4 hours. The total number of caps produced is 6000. If the older machine worked for 4 hours and the newer machine for 2 hours, the total produced would be 5000. How many caps does each machine produce in an hour?

(m) A man invests two sums of money, one sum at 3% and one sum at 5%. The total income is \$370. If each amount were invested at 4%, the total income would be \$360. What is the total amount invested?